The supplier–buyer integrated production–inventory model with random yield, scrapping, and discounting sales

by

Sreesuryasai Santosh Sarma Peesapati

B.Tech. (Mechanical Engineering)
Jawaharlal Nehru Technological University,
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Abstract

This report provides four models for a single-buyer and single vendor inventory system for a single item to optimize the total cost of this supply chain system. Scrapping imperfect items at buyer's and vendor's locations and discounting imperfect items at buyer's and vendor's locations have been developed. The renewal-reward theorem has been used to calculate the expected total cost and expected replenishment cycle time. Conditions have been provided to adopt a particular model. A numerical example has been provided, sensitivity analysis has been performed to validate the theoretical results, and the input parameters affecting the total cost have been provided.

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Chapter 1

Introduction

Supply chain system coordination between different players of the system has become important for many companies to sustain and grow in the current competitive market. Right from producing a product to delivering the final product to the customer, everything is involved in the supply chain system, and every cost has to be accounted. To grow as a system, the total cost of the system has to be minimal. In the current business environment, it is important to constantly enhance the performance of the inventory systems. Therefore, the integrated vendor-buyer decisions and coordination between them lead to performance improvement and cost optimization (Jaber and Goyal 2008). In the world driven by technology and communication, it is more feasible for coordination and integration in the supply chain system. In literature and practice, it is more convenient to adopt the economic order quantity (EOQ) model to attain the optimal lot size thereby minimizing the total cost of the system. The classical EOQ model assumes that all the items in the system are perfect, but in practical situations, this assumption does not hold well. Therefore, in industries, due to uncontrollable factors, defects occur, and it is impossible to eliminate the defective items and their impact.

In the supply chain system, defects in the items occur because of various factors, which include defective machine setup, human errors, transportation issues, and defective raw material, which is common in any industry (Ullah and Kang 2014). The inspection process is helpful in removing the defective items without being sold to the customers. However, defective items in a shipment result in the buyer being unable to meet the customer's demand. Therefore, defective items in a supply-chain system have to be accounted for and consequent considerations have to be given to meet the customer's demand. Inventory cost of imperfect items and higher production costs influence the total cost of the system (Chen 2017). Coordination between vendor and buyer and integrated decision to optimize the total supply chain cost are more beneficial to both buyer and vendor.

Although the value of the items increases as it goes to the buyer (Abdul-Jalber et al. 2007), it is more meaningful to conduct the inspection process at the buyer's location to eliminate any defects which generally occur due to transportation issues. It is easier for the vendor to conduct the inspection process, so as to initiate the rework or repair (Jamal et al. 2004) and avoid the transportation cost of imperfect items to the vendor (Su 2012). However, it is not practical to ignore the defective items that occur during transportation. As scrapping the defective items or transporting them from the buyer's to the vendor's location always incurs a cost, it is necessary to hold them for a certain number of replenishment cycles so as to optimize the total cost of the inventory system. Holding imperfect items for certain cycles and then either scrapping them at buyer's or vendor's location depending on the cost–effectiveness or discounting them and selling them optimizes the total cost of the supply-chain system. On the contrary, holding imperfect items untill the end of the production cycle always incurs additional holding costs. This is eliminated by calculating the optimal number of cycles to hold the imperfect items.

Defective items in a shipment increase the number of buyer's replenishment cycles as the cycle length varies depending on the percentage of defective items in each shipment, which is a random number. In previous cases, where the defective items were considered as a random number (Salameh and Jaber 2000; Huang 2004), the expected replenishment cycle time was not considered but a fixed cycle time was considered. This leads to a miscalculation of the buyer's holding cost for perfect and imperfect items. Therefore, the total expected cost of the system is improperly estimated. In addition, methods to handle the defective items such as rework or scrapping or selling the items at a discounted price incurs an additional cost, which further increases the total expected cost of the system.

Therefore, in this project, the renewal–reward theorem has been used to calculate the total expected cost of the system and the expected buyer's replenishment cycle time. Methods of handling imperfect items have been discussed in detail to incorporate the additional costs, and solutions have been provided for each method. Sufficient comparisons were made between the models to adopt a particular model depending on the condition satisfied.

Chapter 2

Literature Review

The basic model of economic order quantity assumes that all the products in each replenishment cycle are perfect (Banerjee 1986). In practicality, it is impossible to have all the items in perfect quality and some imperfections might arise due to machine break-down, human mistakes or transportation issues. As a result of these imperfect items in each replenishment cycle, the buyer might face a stock-out or should order more frequently than actually planned. The imperfect items should either be disposed of or sold at a discounted price or should be reworked which results in additional cost. Therefore, as to decrease the total cost of the system, the percentage of defective items in each shipment and their handling ways have to be taken into account.

The presence of defective items in inventory systems has been extensively investigated, considering the probability of defective items as a random variable. Shih (1980) considered the presence of defective items in the inventory system which results in shortages, and a model was developed where defective items are considered as a random variable. Zhang and Gerchak (1990) was one among the first to consider random yield in the EOQ model in which defective items have not been replaced and have to be replaced. Salameh and Jaber (2000) provided an extension to the basic EOQ model by considering the items of imperfect quality. The percentage of imperfect items was considered as a random number. However, the time interval between successive shipments was not considered as a random variable. Khouja (2003) considered a single-vendor and single-buyer system with an assumption that all the imperfect items are reworked at a certain cost. However, this assumption neglected the stochastic nature of the demand-supply system and induced the flaw that the time is not considered as a random variable. Huang (2004) extended the model of Salameh and Jaber (2000) however did not consider buyer's random cycle time similar to Salameh and Jaber (2000). Consequently, the stochastic nature of the supplier's inventory profile and buyer's random cycle times were neglected in Huang (2004) . Salameh and Jaber (2000) and Huang (2004) papers only considered optimizing the buyer's cost. An extension of these papers was given by Kelle et al. (2009) and Chen and Kang (2010) where there is coordination between the buyer and the supplier. However, as these papers were extensions of Huang (2004), the basic flaw was still not corrected in this work. Maddah and Jaber (2008) revisited the model developed by Salameh and Jaber (2000) to rectify the flaw that the order quantity is greater than the order quantity in the EOQ model when the yield variability is low. Wahab and Jaber (2010) also provided a note on Salameh and Jaber (2000) different holding costs for good and defective items when there are learning effects. Goyal and Cárdenas-Barrón (2002) suggested a simple model for the model proposed by Salameh and Jaber (2000). Other works on imperfect items include works by Liao et al. (2018), Aslani et al. (2017), and Teimoori et al. (2019). Liao et al. (2018) proposed a profit-maximizing EOQ model in which imperfect items were produced during production. However, the random yield has not been considered and handling the imperfect items also has not been discussed. Aslani et al. (2017) proposed two strategies to improve the mean and to reduce the variability of yield in an EOQ model with random yield. This model lacks the coordination between buyer and vendor and ways to handle imperfect items has not been considered. Teimoori et al. (2019) proposed a new inventory model managed by the vendor where the defective items are screened at the buyer's location. However, the model did not account for the handling costs of the defective items.

Furthermore, handling imperfect items gained prominence in recent years. Lin (2010) proposed an inventory model in which the supplier provides a compensational discount on the imperfect items to counter holding costs of imperfect items at the buyer's location. Cheikhrouhou et al. (2018) discussed two models for sending the imperfect items after inspection at the buyer's location to the supplier's location but did not propose discarding or reworking the imperfect items. Taleizadeh et al. (2016) proposed a model where the imperfect items are produced during the production period only and a reworking process eliminates these imperfections to make them perfect items to meet the demand. As defects occur due to human errors or transportation issues or other factors and reworking of all imperfect items is difficult in a practical case, the model is not applied to all the cases. Pal and Mahapatra (2017) modeled a three-layer supply chain system with a manufacturer, supplier, and retailer. After inspection of raw material at the manufacturer's location, defective raw materials are returned to the supplier and imperfect items are reworked. The imperfect items are produced when the machine is unreliable which follows a density function. The integrated inventory system model assumes all the imperfect items as defect–free after rework which is not possible for all the cases. Khan et al. (2014) provided a simple integrated model for a single vendor-single buyer inventory system where the quality inspection errors at the buyer's location are taken into account. These inspection errors might result in adverse effects including supplying defective items to customers or rejecting perfect items as defective items. Thus, defective items play an important role in calculating the total cost of the system. Ullah and Kang (2014) proposed an inventory model by considering imperfect production and inspection. The effects of rework and rejections were discussed. Taleizadeh et al. (2014) proposed a model to rework imperfect items, another model to discuss selling the imperfect items at a discounted price and another model to scrap the imperfect items. Brauer and Buscher (2018) identified the shortcomings of Taleizadeh et al. (2016) where the holding costs of manufacturer and retailer were incorrectly calculated. Hsu and Hsu (2013) developed an EOQ model with imperfect items, shortages, backorders and returns. Lin and Lin (2018) investigated an inventory model with defective items in which the buyer inspects the incoming delivery and the vendor offers a discount on the defective items depending on the defective rate. The model optimizes the total profit of the system. Chen (2017) extended the EOQ model by considering imperfect items and items with deteriorating quality. Effects of replenishment, the rework of deteriorating items were investigated. Cheng et al. (2018) proposed a vendor-buyer inventory model with defective items and salvage strategies which optimizes the total cost of the system. Liu and Çetinkaya (2011) investigated the model considering the expected total cost of the system in the case of random yield. Hence, summarizing all the above works, defective items and their handling impacts the total cost of the inventory system. Holding cost of imperfect items, change in the replenishment cycle of the buyer to avoid shortages and adopting a suitable method to either scrap, discount or rework the defective items depending on the condition of them is important in practical scenarios.

In this project, a single-vendor and single buyer inventory system with one item has been considered. Two models to scrap the imperfect item at the buyer's and vendor's location and two models to sell the defective items at a discounted price at buyer's location and vendor's location have been discussed in detail. To accurately capture the replenishment cycles, expected replenishment cycle times have been considered and the renewal-reward theorem has been applied to calculate the total expected supply chain system. In most of the models, defective items have been held till the end of the production cycle to either dispose or rework them which increases their holding cost and thereby increasing the total cost of the system. A new method has been proposed in the following section to optimize the cost of defective items, thereby, optimizing the total cost of the system.

Chapter 3

The Model

In this section, four different cases are modeled for a single vendor and single buyer with one product, which has defects in each shipment, to minimize the total cost of the integrated supply chain system. The models use the renewal-reward theorem (Sheldon 2010) to calculate the annual expected cost. The notations used for the modeling are given below in Table 3.1.

The vendor manufactures at a finite rate of R and ships them in equally sized batches of quantity Q to the buyer and the number of shipments from the vendor to the buyer n. Shortages are not allowed. As there are defective items in each shipment, there is a probability that the received quantity might not complement the quantity as requested by the buyer. Inspection occurs at the buyer's location as the defects might occur due to improper transportation as well. Assuming that p_i is the percentage of perfect items in shipment i, and for each shipment i, p_i is independent. Also, we assume $\mathbb{E}[p_i] = \mu$ and $Var[p_i] = \delta^2$. As the production rate is very high, the vendor has enough inventory to fulfill the buyer's demand within each shipment. Therefore, we make an assumption to ensure that the vendor has enough inventory to meet the buyer's demand. Therefore, $p_i R > D$. If this condition is satisfied, for every p_i, the vendor is able to meet the demand of the buyer for every shipment. As discussed earlier four models are presented,

Table 3.1: Notations to be used in mathematical models

s_v	Setup cost of vendor per production run
s_b	Setup cost of buyer per each order
h_{bp}	Buyer's holding cost for perfect items delivered per item
h_v	Vendor's holding cost for inventory per unit time per each item
h_{vi}	Vendor's holding cost for imperfect items per unit time per each item
d_b	Salvage cost of imperfect items at buyer's location per each item
d_v	Salvage cost of imperfect items at vendor's location per each item
c_v	Vendor's cost for production of each item
c_b	Buyer's cost for items from vendor per each item
c_{bi}	Buyer's cost for inspection of each item
c_{bl}	Buyer's cost to sell the imperfect items at a lower price
c_{vl}	Vendor's cost to sell the imperfect items at a lower price
D	Annual demand
R	Vendor's production rate ($R > D$)
p_i	Percentage of perfect items in *ith* delivery
d_{bi}	Fixed salvage cost for imperfect items at buyer's location
d_{vi}	Fixed salvage cost for imperfect items at vendor's location
t_v	Fixed transport cost for imperfect items from buyer's location to vendor's location
t_{vi}	Transportation cost of each imperfect item from buyer to vendor's location
$\mathbb{E}[T]$	Buyer's expected cycle time
Q	Buyer's order quantity per cycle
Q_p	Vendor's total production per production run
T_p	Time required for vendor for production of Q_p items
n	Number of shipments per production run from vendor to buyer per production run
N	Number of shipments buyer holds the imperfect items
$\mathbb{E}[TC(Q,n,N)_b]$	Buyer's expected average cost per unit time
$\mathbb{E}[TC(Q,n,N)_v]$	Vendor's expected average cost per unit time
$\mathbb{E}[TC(Q,n,N)]$	Supply chain's expected average cost per unit time

namely, scrapping imperfect items at the buyer's location, discounting imperfect items at the buyer's location, scrapping imperfect items at the vendor's location and discounting imperfect items at the vendor's location.

In the case where imperfect items are scrapped at the vendor's location and buyer's location, there is a fixed cost each time a salvage occurs and in addition, there is salvage cost per each item. Therefore, imperfect items are stored for N cycles and then scrapped. For the case of scrapping in the vendor's location, as the inspection takes place in the buyer's location, there is transportation cost of imperfect items from the buyer's location to the vendor's location which are discussed in detail in the following sections.

For the case of discounting imperfect items where item damage is minimal, the defective items are sold at a discounted price, but these items do not serve the buyer's demand. As discussed, there are two cases where the defective items are sold at the buyer's place and defective items sold at the vendor's place. For the case in the buyer's place, the items are sold at the end of each replenishment which is more practical. But considering the case in the vendor's place, the defective items are stored for N cycles as there is a transport cost of imperfect items from the vendor's location to the buyer's location. All four cases are discussed in detail in the following sections. As there is a defective percentage of imperfect items in each shipment, the replenishment cycle time is an expected value and not the same for each shipment. And more importantly, the total cost of the supply chain system is also an expected value. As discussed, the expected total annual cost is calculated by using the renewal-reward theorem.

$$\mathbb{E}[TC(Q,n,N)] = \frac{\mathbb{E}[\text{Total cost of system}]}{\mathbb{E}[\text{Total cycle time}]}. \tag{3.1}$$

Let $T_i = \frac{p_i Q}{D}$ for $i = 1$ to n. As mentioned, p_i is an independent variable, T_i is also an independent variable which gives us the buyer's replenishment's length. Also, $K_0 = 0$, $K_i = \sum_1^i T_i$. We also consider a counting process which counts the number of buyer's replenishment, by time t. $A(t) = Ni; K_i \leq t$. Z_i is the i^{th} fold convolution. Therefore, we get,

$$P\{A(t) \geq i\} = Z_i \left(\frac{T_i D}{Q} \right). \tag{3.2}$$

3.1 Expected total cycle length

The buyer's replenishment cycle time varies for each shipment because of the percentage of defective items in each shipment. As soon as the buyer's inventory ends, the supplier ships the buyer's requested quantity making sure that there are no shortages. As we know that there are n shipments in each cycle, the expected total cycle length are given by

$$\begin{aligned} \mathbb{E}[\text{Total cycle length}] &= \mathbb{E}\left[\sum_{i=1}^{n} T_i\right], \\ &= \frac{Q}{D}\mathbb{E}[\sum_{i=1}^{n} p_i], \\ &= \frac{n\mu Q}{D}. \end{aligned} \tag{3.3}$$

3.2 Scrapping imperfect items at the buyer's location

In this case, the vendor sends quantity Q to the buyer, and the buyer inspects them at the buyer's location as defects might occur in the transportation of items as well. After inspection, the buyer holds the imperfect items for N cycles and disposes them at the buyer's location. There is a fixed cost of d_{bi} for each salvage and variable cost of d_b for each item during salvage. The various costs involved are given below.

3.2.1 Buyer's expected total cost function

The buyer's cost function has the buyer's setup cost, inspection cost, and holding cost of perfect items.

Buyer's setup cost:

The buyer has to pay a fixed cost of s_b per each order. Therefore, the buyer's setup cost per unit time is given by

$$\begin{aligned} S_b &= \frac{s_b}{\mathbb{E}[T]}, \\ &= \frac{s_b D}{\mu Q}. \end{aligned} \tag{3.4}$$

Buyer's inspection cost:

Inspection is done at the buyer's location to segregate the defective items from the perfect items. Inspection is performed at a cost of c_{bi} per each item. Therefore, the buyer's inspection cost per unit time is given by

$$\begin{aligned} C_{bi} &= \frac{Q c_{bi}}{\mathbb{E}[T]}, \\ &= \frac{c_{bi} D}{\mu}. \end{aligned} \tag{3.5}$$

Buyer's holding cost of perfect items:

After inspection, the buyer holds the perfect items until the end of each replenishment cycle. Therefore, the average holding cost of perfect items per unit time is given by

$$\begin{aligned} H_{bp} &= h_{bp}\mathbb{E}[\sum_{i=1}^{n} \frac{Q^2 p_i^2}{2D}]/n\mathbb{E}[T], \\ &= h_{bp}\frac{Q(\mu^2+\delta^2)}{2\mu}. \end{aligned} \tag{3.6}$$

Expected total cost of the buyer:

From Equations (3.4), (3.5), and (3.6), the expected total cost of the buyer is given by

$$\mathbb{E}[TC(Q,n,N)_b] = \frac{s_b D}{\mu Q} + \frac{c_{bi} D}{\mu} + h_{bp}\frac{Q(\mu^2+\delta^2)}{2\mu}. \tag{3.7}$$

3.2.2 Vendor's expected total cost function

As the vendor is responsible for the imperfect items, the holding cost of imperfect items held at the buyer's location is paid by the vendor. The vendor's cost function involves setup cost, manufacturing cost, holding cost of items at the vendor's place, holding cost of imperfect items at the buyer's place, and scrapping of imperfect items at the buyer's location.

Vendor's setup cost:

The vendor has a setup cost of s_v for each production cycle. Therefore, the vendor's fixed setup cost per year is given by

$$\begin{aligned} S_v &= s_v/n\mathbb{E}[T], \\ &= \frac{s_v D}{n\mu Q}. \end{aligned} \tag{3.8}$$

Vendor's manufacturing cost:

The vendor manufactures items at the cost of c_v for each item. Therefore, the vendor's manufacturing cost per year is given by

$$\begin{aligned} C_v &= c_v Q/\mathbb{E}[T], \\ &= \frac{c_v D}{\mu}. \end{aligned} \tag{3.9}$$

Vendor's expected holding cost of imperfect items at the buyer's location:

The vendor pays the holding cost of imperfect items that are stored for N cycles at the buyer's location before they are either scrapped or returned to the vendor's location.

$$\begin{aligned} H_{vi} &= h_{vi}QE[T]\{[(1-p_1)] + [(1-p_1)+(1-p_2)] \\ &\quad + ...[(1-p_1)+(1-p_2)...+(1-p_N)]\}/NE[T], \\ &= \frac{h_{vi}Q\mathbb{E}[T]}{N\mathbb{E}[T]}\frac{N(N+1)}{2} - \sum_{k=1}^{N-k+1} p_k. \end{aligned} \tag{3.10}$$

Now, to calculate the expected holding cost of imperfect items,

$$\begin{aligned}
\mathbb{E}[H_{vi}] &= \frac{h_{vi}Q}{N}\{\mathbb{E}[\frac{N(N+1)}{2} - \sum_{k=1}^{N-k+1} p_k]\}, \\
&= \frac{h_{vi}Q}{N}\left[\frac{N(N+1)}{2} - \mu\{N^2 - \frac{N(N+1)}{2} + N\}\right], \\
&= \frac{h_{vi}Q}{N}\left[\frac{N(N+1)}{2} - \mu\{\frac{N(N+1)}{2}\}\right], \\
&= \frac{h_{vi}Q(1-\mu)(N+1)}{2}. \qquad (3.11)
\end{aligned}$$

Holding cost of items at the vendor's location:

The vendor's inventory profile is a stochastic process because the time between each shipment is a random number. As a result, the number of imperfect items in each shipment is also a random number. Let the vendor's inventory be denoted by H_v.

Let

$$\phi = \frac{(n-1)Q}{R}. \qquad (3.12)$$

$$T = \sum_{i=1}^{n} Y_i. \qquad (3.13)$$

According to Equation (3.12), ϕ is the time required to complete the production of a lot after the initial shipment and according to Equation (3.13), T is the total length of the production.

Let H_0 be the cumulative inventory at the supplier's location between the start of the production of a batch and the first shipment to the vendor. This is a deterministic case and is given by

$$H_0 = \frac{Q^2}{2R}. \qquad (3.14)$$

Let H_1 be the cumulative inventory held at the vendor's location between the first shipment to the vendor and the first shipment immediately after the end of the production by the vendor. It is hard to compute it directly as this is a stochastic process. Therefore, we calculate it in the following steps. Let H be total inventory that is accumulated right after the first shipment till the end of the buyer's last replenishment cycle if there were

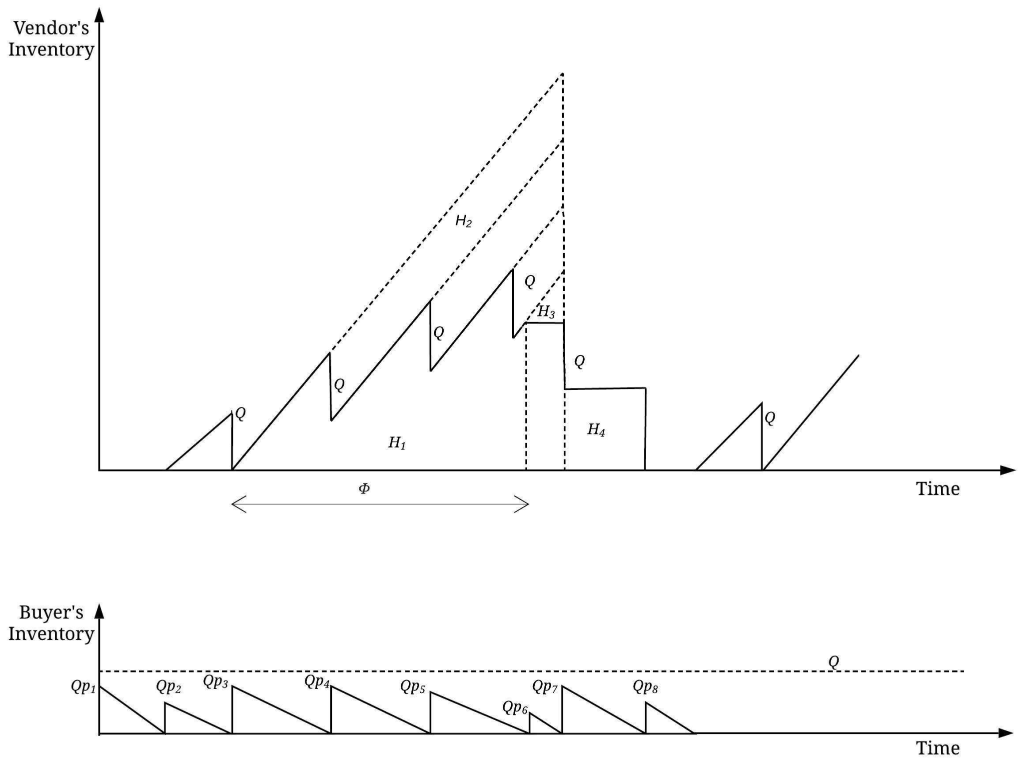

Figure 3.1: A realisation of supplier's and buyer's inventory profiles

no replenishment in between. From Figure 3.1, the base and height of the big triangle H are given by $\frac{Q}{D}\sum_{i=1}^{N(\phi)+1} p_i$, and $\frac{QR}{D}\sum_{i=1}^{N(\phi)+1} p_i$ respectively,

$$H = \frac{Q^2 R}{2D^2}(\sum_{i=1}^{N(\phi)+1} p_i)^2. \tag{3.15}$$

Let H_2 be the excess cumulative inventory at the vendor's location if the vendor produces after the first shipment till the time for the first shipment after ϕ without any intermediate shipments. Let H_3 be the accumulated inventory if the supplier produces after ϕ till the first shipment after ϕ. If there are 2 or less than 2 shipments, $N(\phi) = 0$ as there are no shipments between the first shipment and the end of production. From Figure 3.1, H_2 is a set of parallelograms. Therefore,

$$
\begin{aligned}
H_2 &= Q\frac{Q}{D}\{(p_2+p_3+....+p_{N(\phi)+1})+(p_3+p_4+..+p_{N(\phi)+1})+...+(p_{N(\phi)+1})\},\\
&= Q\frac{Q}{D}\{(\sum_{j=2}^{N(\phi)+1} p_j)+(\sum_{j=3}^{N(\phi)+1} p_j)+...+(\sum_{j=N(\phi)+1}^{N(\phi)+1} p_j)\},\\
&= \frac{Q^2}{D}\{\sum_{i=1}^{N(\phi)}(\sum_{j=i+1}^{N(\phi)+1} p_j)\}. \qquad (3.16)
\end{aligned}
$$

As we know that H_2 is zero if $n=2$, and following Equation (3.16), we have.

$$
H_2 = \begin{cases} 0, & \text{if } n=2 \\ \frac{Q^2}{D}\{\sum_{i=1}^{N(\phi)}(\sum_{j=i+1}^{N(\phi)+1} p_j)\}. & \text{if } n>2 \end{cases} \qquad (3.17)
$$

H_3 is the accumulated inventory if the supplier produces after ϕ till the first shipment after ϕ. From Figure 3.1, H_3, is the triangle with base and height given by $\frac{Q}{D}(\sum_{i=1}^{N(\phi)+1} p_i-\phi)$ and $\frac{QR}{D}(\sum_{i=1}^{N(\phi)+1} p_i-\phi)$. Therefore, H_3 is given by

$$
\begin{aligned}
H_3 &= \frac{Q}{D}(\sum_{i=1}^{N(\phi)+1} p_i-\phi)\frac{QR}{D}(\sum_{i=1}^{N(\phi)+1} p_i-\phi),\\
&= \frac{Q^2R}{D^2}(\sum_{i=1}^{N(\phi)+1} p_i-\phi)^2. \qquad (3.18)
\end{aligned}
$$

To calculate the cumulative inventory held at the vendor's location between the first shipment to the buyer and the first shipment immediately after the end of the production by the vendor, H_1, from Figure 3.1, to obtain the area under H_1, we subtract H_2+H_3 from H. Therefore, $H_1 = H - H_2 - H_3$.

$$
H_1 = \begin{cases} \phi RQp_i/D - \phi^2R/2, & \text{if } n=2 \\ \frac{QR\phi}{D}(\sum_{i=1}^{N(\phi)+1} p_i) - \phi^2R/2 - \frac{Q^2}{D}\sum_{i=1}^{N(\phi)}(\sum_{j=i+1}^{N(\phi)+1} p_j). & \text{if } n>2 \end{cases} \qquad (3.19)
$$

As $\phi+1$ is the stopping time for production, $N(\phi)+1$ is the stopping time for p_i. Using Wald's equation (Sheldon 2010), we calculate the expected cost of H_1 Therefore,

$$
\mathbb{E}[\frac{\phi QR}{D}\sum_{i=1}^{N(\phi)+1} p_i] = \frac{\phi\mu QR}{D}\mathbb{E}[N(\phi)+1]. \qquad (3.20)
$$

Let

$$U_i = \begin{cases} 1, & \text{if } N(\phi) + 1 \geq i, \\ 0. & \text{otherwise }. \end{cases} \tag{3.21}$$

As we know that U_i and p_i are independent because of the fact that $N(\phi)+1$ is stopping time for p_i, we have

$$\begin{aligned} \mathbb{E}[\frac{Q^2}{D}\sum_{i=1}^{N(\phi)}(\sum_{j=i+1}^{N(\phi)+1} p_j)] &= \frac{Q^2}{D}\mathbb{E}[\sum_{i=2}^{N(\phi)+1}(i-1)p_i], \\ &= \frac{Q^2}{D}\mathbb{E}[\sum_{i=2}^{n-1}(i-1)p_iU_i], \\ &= \frac{Q^2}{D}\sum_{i=2}^{n-1}(i-1)\mathbb{E}[p_i]\mathbb{E}[U_i], \\ &= \frac{\mu Q^2}{D}\sum_{i=2}^{n-1}(i-1)P\{N(\phi)+1 \geq i\}, \\ &= \frac{\mu Q^2}{D}\sum_{i=1}^{n-2} iP\{N(\phi) \geq i\}. \end{aligned} \tag{3.22}$$

Therefore $\mathbb{E}[H_1]$ is given by,

$$\mathbb{E}[H_1] = \begin{cases} \phi\mu QR/D - \phi^2 R/2, & \text{if } n = 2, \\ \frac{\phi\mu QR}{D}\mathbb{E}[N(\phi)+1] - \phi^2 R/2 - \frac{\mu Q^2}{D}\sum_{i=1}^{n-2} iP\{N(\phi) \geq i\}. & \text{if } n > 2 \end{cases} \tag{3.23}$$

Let H_4 be the cumulative inventory at the supplier's location between the first shipment just after the end of production and the last shipment of batch.

To compute the expected value of H_4, we use the renewal theory where $N(t)$ registers the number of shipments by the time t.

1. If $N(\phi) = n - 1$, then as ϕ is the time required to complete the production of a lot after the initial shipment, there are no shipments after the entire production. Therefore, $H_4 = 0$ as the total shipments in the production cycle are n.

2. If $N(\phi) = n - 2$, there is only one shipment after the production of the lot and that one shipment is covered under the inventory calculation as per the definition of H_2. Therefore again, $H_4 = 0$.

3. If $N(\phi) = n - 3$, there is one shipment and if $N(\phi) <= n - 3$, then the number of shipments associated with H_4 is given by $n - 2 - N(\phi)$.

H_4 is given by,

$$\begin{aligned} H_4 = Q(n-2-N(\phi))\frac{Qp_{\{N(\phi)+2\}}}{D} + Q(n-3-N(\phi))\frac{Qp_{\{N(\phi)+3\}}}{D} \\ + ... + Q(1)\frac{Qp_{\{n-1\}}}{D}, \\ = \frac{Q^2}{D}\sum_{i=2}^{n-1-N(\phi)} [n-i-N(\phi)]p_{\{N(\phi)+1\}}. \end{aligned} \tag{3.24}$$

Thus,

$$H_4 = \begin{cases} 0, & \text{if } N(\phi) > n-2, \\ \frac{Q^2}{D}\sum_{i=2}^{n-1-N(\phi)}[n-i-N(\phi)]p_{\{N(\phi)+1\}}. & \text{if} N(\phi). < n-2 \end{cases} \tag{3.25}$$

Now to calculate the expected value of H_4 we use conditional probability for $n \geq 3$, $N(\phi)$ and $p\{N(\phi)+1\}$ are independent terms and we use conditional probability to calculate the expected value of H_4.

$$\begin{aligned} \mathbb{E}[H_4] = \mathbb{E}[\mathbb{E}[H_4/N(\phi)]] = \frac{Q^2}{D}\sum_{k=0}^{n-3}\mathbb{E}\left(\sum_{i=2}^{n-1-k}[n-i-k]p_{i+k}\right)P\{N(\phi)=k\}, \\ = \frac{\mu Q^2}{D}\sum_{k=0}^{n-3}\sum_{i=2}^{n-1-k}[n-i-j]P\{N(\phi)=k\}, \\ = \frac{\mu Q^2}{2D}\sum_{k=0}^{n-3}(n-1-j)(n-2-j)P\{N(\phi)=k\}. \end{aligned} \tag{3.26}$$

The expected total holding cost of the supplier is given by $\mathbb{E}[H_v] = h_v(H_0+\mathbb{E}[H_1]+\mathbb{E}[H_4])$. It has to be observed that H_0 is a deterministic value and H_1 and H_4 are expected values.

Therefore, $\mathbb{E}[H_v]$ is given by,

$$\mathbb{E}[H_v] = \begin{cases} h_s(\frac{Q^2}{2R} + \phi\mu QR/D - \phi^2 R/2), & \text{if } n = 2 \\ h_s(\frac{Q^2}{2R} + \frac{\phi\mu QR}{D}\mathbb{E}[N(\phi)+1] - \phi^2 R/2 - \frac{\mu Q^2}{D}\sum_{i=1}^{n-2} iP\{N(\phi) \geq i\}) & \\ +h_s(\frac{\mu Q^2}{2D}\sum_{k=0}^{n-3}(n-1-j)(n-2-j)P\{N(\phi) = k\}). & \text{otherwise} \end{cases} \tag{3.27}$$

When $n \geq 3$, following the convolution function, we get

$$\phi = \frac{(n-1)Q}{R}.$$

By substituting the above equation in Equation (3.2), we get

$$\begin{aligned} \mathbb{E}N(\phi) &= Z_i(\frac{\phi D}{Q}), \\ &= Z_i(\frac{(n-1)D}{R}). \end{aligned} \tag{3.28}$$

$$\mathbb{E}[N(\phi)+1] = 1 + \sum_{i=2}^{n-2} Z_i(\frac{(n-1)D}{R}). \tag{3.29}$$

$$P\{N(\phi) \geq i\} = Z_i(\frac{(n-1)D}{R}). \tag{3.30}$$

$$P\{N(\phi) = i\} = Z_i(\frac{(n-1)D}{R}) - Z_{i+1}(\frac{(n-1)D}{R}). \tag{3.31}$$

Let

$$C = \frac{(n-1)D}{R}$$

.

By substituting Equations (3.28), (3.29), (3.30), and (3.31) in Equation (3.27), we get

$$
\begin{aligned}
\mathbb{E}[H_v] &= h_v[\frac{Q^2}{2R} + \frac{\mu(n-1)Q^2}{D} - \frac{(n-1)^2Q^2}{2R} + \frac{\mu Q^2}{D}\sum_{i=1}^{n-2}(n-1-i)Z_i(C) \\
&+ \frac{\mu Q^2}{2D}\sum_{i=0}^{n-3}(n-1-i)(n-2-i)[Z_i(C) - Z_{i+1}(C)]], \\
&= h_v[\frac{Q^2}{2R} + \frac{\mu(n-1)Q^2}{D} - \frac{(n-1)^2Q^2}{2R} + \frac{\mu Q^2}{D}\sum_{i=1}^{n-2}(n-1-i)Z_i(C) \\
&+ \frac{\mu Q^2}{2D}\sum_{i=0}^{n-3}(n-1-i)(n-2-i)Z_i(C) - \frac{\mu Q^2}{2D}\sum_{i=1}^{n-2}(n-i)(n-1-i)Z_i(C)], \\
&= h_v[\frac{Q^2}{2R} + \frac{\mu(n-1)Q^2}{D} - \frac{(n-1)^2Q^2}{2R} + \frac{\mu Q^2}{D}(n-1)(n-2)], \\
&= h_v[\frac{\mu Q^2 n(n-1)}{2D} - \frac{Q^2 n(n-2)}{2R}]. \qquad (3.32)
\end{aligned}
$$

Therefore expected holding cost per unit time is given by

$$
\begin{aligned}
\frac{\mathbb{E}[H_v]}{\mathbb{E}[T]} &= h_v[\frac{\mu Q^2 n(n-1)}{2D} - \frac{Q^2 n(n-2)}{2R}]/\{n\mu Q/D\}, \\
&= h_v[\frac{Q(n-1)}{2} - \frac{QD(n-2)}{2\mu R}]. \qquad (3.33)
\end{aligned}
$$

Salvage cost of imperfect items:

The buyer holds them for N cycles and salvages them at fixed salvage cost of d_{bi} for every salvage and a cost of d_b per each item. Therefore, the salvage cost of imperfect items at the buyer's location per year is given by

$$
\begin{aligned}
D_b &= \mathbb{E}\sum_{i=1}^{N}(1-p_i)Qd_b + d_{bi}/N\mathbb{E}[T], \\
&= \{(N - N\mu)Qd_b + d_{bi}\}D/N\mu Q, \\
&= \frac{\{N(1-\mu)Qd_b + d_{bi}\}D}{N\mu Q}. \qquad (3.34)
\end{aligned}
$$

Vendor's expected total cost:

From Equations (3.8), (3.9), (3.10), (3.33), and (3.34), the expected total cost of the vendor is given by

$$\begin{aligned}\mathbb{E}[TC(Q,n,N)]_v = {} & \frac{s_v D}{n\mu Q} + \frac{c_v D}{\mu} + h_v[\frac{Q(n-1)}{2} - \frac{QD(n-2)}{2\mu R}] \\ & + \frac{h_{vi} Q(1-\mu)(N+1)}{2} + \frac{\{N(1-\mu)Qd_b + d_{bi}\}D}{N\mu Q}.\end{aligned} \tag{3.35}$$

3.2.3 Expected total cost of supply chain system

The expected total cost of the system is given by Equations (3.7), and (3.35). Therefore, the total cost of the supply chain is given by

$$\begin{aligned}\mathbb{E}[TC(Q,n,N)] = {} & \frac{s_v D}{n\mu Q} + \frac{s_b D}{\mu Q} + \frac{c_v D}{\mu} + \frac{c_{bi} D}{\mu} \\ & + h_v[\frac{Q(n-1)}{2} - \frac{QD(n-2)}{2\mu R}] + h_{bp}\frac{Q(\mu^2+\delta^2)}{2\mu} \\ & + \frac{h_{vi} Q(1-\mu)(N+1)}{2} + \frac{\{N(1-\mu)Qd_b + d_{bi}\}D}{N\mu Q}.\end{aligned} \tag{3.36}$$

3.2.4 Theoretical solutions

To calculate the optimal cost of the system, we partial differentiate the expected total cost Equation (3.36) with respect to Q, n and N, equate it to zero and find the optimal Q, n and N and substitute them in the expected total cost function.

By differentiating Equation (3.36) with respect to Q, we get

$$\begin{aligned}\frac{\partial E[TC(Q,n,N)]}{\partial Q} = {} & \frac{-s_v D}{n\mu Q^2} - \frac{s_b D}{\mu Q^2} + \frac{h_v[n(\mu R - D) - (\mu R - 2D)]}{2\mu R} + \frac{h_{bp}(\mu^2+\delta^2)}{2\mu} \\ & + \frac{h_{vi}(1-\mu)(N+1)}{2} - \frac{d_{bi} D}{N\mu Q^2}.\end{aligned} \tag{3.37}$$

By equating the above Equation (3.37) to zero, we get

$$\frac{2DR(Ns_v + nNs_b + nd_{bi})}{nNQ^2} = h_v[n(\mu R - D) - (\mu R - 2D)] + h_{bp}R(\mu^2 + \delta^2) + h_{vi}R\mu(1-\mu)(N+1).$$

$$Q = \sqrt{\frac{2DR(Ns_v + nNs_b + nd_{bi})}{nN\left[h_v\{n(\mu R - D) - (\mu R - 2D)\} + h_{bp}R(\mu^2 + \delta^2) + h_{vi}R\mu(1-\mu)(N+1)\right]}}. \tag{3.38}$$

By differentiating Equation (3.36) with respect to n, we get

$$\frac{\partial E[TC(Q,n,N)]}{\partial n} = \frac{-s_v D}{n^2 \mu Q} + \frac{h_v Q(\mu R - D)}{\mu R}. \tag{3.39}$$

By equating the above Equation (3.39) to zero, we get

$$n = \frac{1}{Q}\sqrt{\frac{2s_v DR}{h_v(\mu R - D)}}. \tag{3.40}$$

By differentiating Equation (3.36) with respect to N, we get

$$\frac{\partial E[TC(Q,n,N)]}{\partial N} = \frac{-d_{bi} D}{N^2 \mu Q} + \frac{h_{vi} Q(1-\mu)}{2}. \tag{3.41}$$

By equating the above Equation (3.41) to zero, we get

$$N = \frac{1}{Q}\sqrt{\frac{2d_{bi} D}{h_{vi}\mu(1-\mu)}}. \tag{3.42}$$

By substituting Equations (3.40), and (3.42) in Equation (3.38) and squaring it we get,

$$Q^2\left[h_{bp}R(\mu^2+\delta^2) + h_{vi}R\mu(1-\mu)\right] - Q^2\left[h_v(\mu R - 2D)\right] + Q\left[\sqrt{2DRh_v s_v(\mu R - D)}\right] + Q\left[R\sqrt{2Dd_{bi}h_{vi}\mu(1-\mu)}\right] = 2DRs_b + Q\left[\sqrt{2DRh_v s_v(\mu R - D)}\right] + QR\left[\sqrt{2Dd_{bi}h_{vi}\mu(1-\mu)}\right].$$

$$Q = \sqrt{\frac{2s_b DR}{h_{bp}R(\mu^2+\delta^2) + h_{vi}R\mu(1-\mu) - h_v(\mu R - 2D)}}. \tag{3.43}$$

By substituting Equation (3.43) in Equations (3.40) and (3.42), we get

$$n = \sqrt{\frac{s_v[h_{bp}(\mu^2+\delta^2) + h_{vi}R\mu(1-\mu) - h_v(\mu R - 2D)]}{s_b h_v(\mu R - D)}}. \tag{3.44}$$

$$N = \sqrt{\frac{d_{bi}[h_{bp}(\mu^2+\delta^2) + h_{vi}R\mu(1-\mu) - h_v(\mu R - 2D)]}{s_b h_{vi}R\mu(1-\mu)}}. \tag{3.45}$$

3.2.5 Proof of convexity

To prove that the obtained values Q, n and N, are optimal, we consider the Hessian matrix of second-order partial derivatives of the expected value of the total cost of the supply chain system.

Second-order partial derivative with respect to Q, n and N for the partial derivative of expected total cost with respect to Q.

$$\frac{\partial^2 E[TC(Q,n,N)]}{\partial^2 Q} = \frac{2s_v D}{n\mu Q^3} + \frac{2s_b D}{\mu Q^3} + \frac{2d_{bi}D}{N\mu Q^3}. \tag{3.46}$$

$$\frac{\partial^2 E[TC(Q,n,N)]}{\partial Q \partial n} = \frac{s_v D}{n^2\mu Q^2} + \frac{h_v}{2}(1 - \frac{D}{\mu R}). \tag{3.47}$$

$$\frac{\partial^2 E[TC(Q,n,N)]}{\partial Q \partial N} = \frac{d_{bi} D}{N^2\mu Q^2} + \frac{h_{vi}}{2}(1 - \mu). \tag{3.48}$$

Second-order partial derivative with respect to Q, n and N for the partial derivative of expected total cost with respect to n.

$$\frac{\partial^2 E[TC(Q,n,N)]}{\partial n \partial Q} = \frac{s_v D}{n^2\mu Q^2} + \frac{h_v}{2}(1 - \frac{D}{\mu R}). \tag{3.49}$$

$$\frac{\partial^2 E[TC(Q,n,N)]}{\partial^2 n} = \frac{2s_v D}{n^3\mu Q}. \tag{3.50}$$

$$\frac{\partial^2 E[TC(Q,n,N)]}{\partial n \partial N} = 0. \tag{3.51}$$

Second-order partial derivative with respect to Q, n and N for the partial derivative of expected total cost with respect to N.

$$\frac{\partial^2 E[TC(Q,n,N)]}{\partial N \partial Q} = \frac{d_{bi} D}{N^2\mu Q^2} + \frac{h_{vi}}{2}(1 - \mu). \tag{3.52}$$

$$\frac{\partial^2 E[TC(Q,n,N)]}{\partial N \partial n} = 0. \tag{3.53}$$

$$\frac{\partial^2 E[TC(Q,n,N)]}{\partial^2 N} = \frac{2d_{bi}D}{N^3 \mu Q}. \tag{3.54}$$

Therefore, the hessian matrix H is given by

$$\begin{bmatrix} \frac{\partial^2 E[TC(Q,n,N)]}{\partial^2 Q} & \frac{\partial^2 E[TC(Q,n,N)]}{\partial Q \partial n} & \frac{\partial^2 E[TC(Q,n,N)]}{\partial Q \partial N} \\ \frac{\partial^2 E[TC(Q,n,N)]}{\partial Q \partial n} & \frac{\partial^2 E[TC(Q,n,N)]}{\partial^2 n} & \frac{\partial^2 E[TC(Q,n,N)]}{\partial n \partial N} \\ \frac{\partial^2 E[TC(Q,n,N)]}{\partial N \partial Q} & \frac{\partial^2 E[TC(Q,n,N)]}{\partial N \partial n} & \frac{\partial^2 E[TC(Q,n,N)]}{\partial^2 N} \end{bmatrix}$$

Let us assume that

$$\begin{bmatrix} \frac{\partial^2 E[TC(Q,n,N)]}{\partial^2 Q} & \frac{\partial^2 E[TC(Q,n,N)]}{\partial Q \partial n} & \frac{\partial^2 E[TC(Q,n,N)]}{\partial Q \partial N} \\ \frac{\partial^2 E[TC(Q,n,N)]}{\partial Q \partial n} & \frac{\partial^2 E[TC(Q,n,N)]}{\partial^2 n} & \frac{\partial^2 E[TC(Q,n,N)]}{\partial n \partial N} \\ \frac{\partial^2 E[TC(Q,n,N)]}{\partial N \partial Q} & \frac{\partial^2 E[TC(Q,n,N)]}{\partial N \partial n} & \frac{\partial^2 E[TC(Q,n,N)]}{\partial^2 N} \end{bmatrix} = \begin{bmatrix} B_1 & B_2 & B_3 \\ B_4 & B_5 & B_6 \\ B_7 & B_8 & B_9 \end{bmatrix} \tag{3.55}$$

The elements in the first principal minor are all positive as given by Equations (3.46), (3.50), and (3.54) and all these values are greater than zero. The elements in the second principal minor are D_1, D_2, and D_3.

$$\begin{aligned} D_1 &= (B_5 B_9) - (B_6 B_8), \\ &= \left(\frac{2s_v D}{n^3 \mu Q} \frac{2d_{bi} D}{N^3 \mu Q} \right) - 0, \\ &= \frac{4D^2 s_v d_{bi}}{n^3 N^3 \mu^2 Q^2}. \end{aligned} \tag{3.56}$$

$$\begin{aligned} D_2 &= (B_1 B_9) - (B_3 B_7), \\ &= \left[\left(\frac{2s_v D}{n \mu Q^3} + \frac{2s_b D}{\mu Q^3} + \frac{2d_{bi} D}{N \mu Q^3} \right) \left(\frac{2d_{bi} D}{N^3 \mu Q} \right) \right] - \left[\left(\frac{d_{bi} D}{N^2 \mu Q^2} + \frac{h_{vi}}{2}(1-\mu) \right)^2 \right]. \end{aligned} \tag{3.57}$$

From Equation (3.42), we know that

$$N = \frac{1}{Q}\sqrt{\frac{2d_{bi}D}{h_{vi}\mu(1-\mu)}}.$$

$$\frac{h_{vi}}{2}(1-\mu) = \frac{d_{bi}D}{N^2\mu Q^2}. \tag{3.58}$$

Substituting Equation (3.58) in Equation (3.57), we get

$$\begin{aligned}
D_2 &= \left[\left(\frac{2s_vD}{n\mu Q^3} + \frac{2s_bD}{\mu Q^3}\right)\left(\frac{2d_{bi}D}{N^3\mu Q}\right)\right] + \left[\left(\frac{2d_{bi}D}{N\mu Q^3}\right)\left(\frac{2d_{bi}D}{N^3\mu Q}\right)\right] \\
&\quad - \left(\frac{d_{bi}D}{N^2\mu Q^2} + \frac{d_{bi}D}{N^2\mu Q^2}\right)^2, \\
&= \left[\left(\frac{2s_vD}{n\mu Q^3} + \frac{2s_bD}{\mu Q^3}\right)\left(\frac{2d_{bi}D}{N^3\mu Q}\right)\right] + \left(\frac{4d_{bi}^2D^2}{N^4\mu^2Q^4}\right) - \left(\frac{4d_{bi}^2D^2}{N^4\mu^2Q^4}\right), \\
&= \left[\left(\frac{2s_vD}{n\mu Q^3} + \frac{2s_bD}{\mu Q^3}\right)\left(\frac{2d_{bi}D}{N^3\mu Q}\right)\right].
\end{aligned} \tag{3.59}$$

$$\begin{aligned}
D_3 &= (B_1B_5) - (B_2B_4), \\
&= \left[\left(\frac{2s_vD}{n\mu Q^3} + \frac{2s_bD}{\mu Q^3} + \frac{2d_{bi}D}{N\mu Q^3}\right)\left(\frac{2s_vD}{n^3\mu Q}\right)\right] - \left[\left(\frac{s_vD}{n^2\mu Q^2} + \frac{h_v}{2}(1-\frac{D}{\mu R})\right)^2\right].
\end{aligned} \tag{3.60}$$

From Equation (3.40), we know that

$$n = \frac{1}{Q}\sqrt{\frac{2s_vDR}{h_v(\mu R - D)}}.$$

$$\frac{h_v}{2}(1-\frac{D}{\mu R}) = \frac{s_vD}{n^2\mu Q^2}. \tag{3.61}$$

Substituting Equation (3.61) in Equation (3.60), we get

$$\begin{aligned}
D_3 &= \left[\left(\frac{2s_bD}{\mu Q^3} + \frac{2d_{bi}D}{N\mu Q^3}\right)\left(\frac{2s_vD}{n^3\mu Q}\right)\right] + \left[\left(\frac{2s_vD}{n\mu Q^3}\right)\left(\frac{2s_vD}{n^3\mu Q}\right)\right] - \left(\frac{s_vD}{n^2\mu Q^2} + \frac{s_vD}{n^2\mu Q^2}\right)^2, \\
&= \left[\left(\frac{2s_bD}{\mu Q^3} + \frac{2d_{bi}D}{N\mu Q^3}\right)\left(\frac{2s_vD}{n^3\mu Q}\right)\right] + \left(\frac{4s_v^2D^2}{n^4\mu^2Q^4}\right) - \left(\frac{4s_v^2D^2}{n^4\mu^2Q^4}\right), \\
&= \left[\left(\frac{2s_bD}{\mu Q^3} + \frac{2d_{bi}D}{N\mu Q^3}\right)\left(\frac{2s_vD}{n^3\mu Q}\right)\right].
\end{aligned} \tag{3.62}$$

All the elements in the second principal minor are also greater than zero. The third principal minor is given by D_4.

$$D_4 = B_1\left[(B_5B_9) - (B_6B_8)\right] - B_2\left[(B_4B_9) - (B_6B_7)\right] - B_3\left[(B_4B_8) - (B_5B_7)\right]. \quad (3.63)$$

By substituting Equations (3.46), (3.47), (3.47), (3.49), (3.50), (3.51), (3.52), (3.53), and (3.54) in Equation (3.63) and solving, we get

$$D_4 = \left(\frac{2s_vD}{n\mu Q^3} + \frac{2s_bD}{\mu Q^3}\right)\left(\frac{2s_vD}{n^3\mu Q}\frac{2d_{bi}D}{N^3\mu Q}\right) + \left(\frac{4s_v^2D^2}{n^4\mu Q^4}\frac{2d_{bi}D}{N^3\mu Q}\right). \quad (3.64)$$

The third principal minor is also a positive term. Since all the principal minors are positive, the values of Q, n and N have been proved to be optimal for the total expected cost of the supply chain system.

3.3 Discounting imperfect items at the buyer's location

In this policy, after inspection of all the items after each delivery, the defective items are held separately and sold at the end of the replenishment cycle at a discounted price per each unit.

3.3.1 Buyer's expected total cost function

The buyer's cost function is the same as the one given in Equation (3.7).

3.3.2 Vendor's expected total cost function

The vendor's cost function involves setup cost, manufacturing cost, holding cost of items at the vendor's location, holding cost of imperfect items at the buyer's location. Let

c_{bl} be the lower price at which imperfect items are sold at the end of each cycle at the buyer's location. And c_b be the price at which the buyer purchases items from the vendor. It has to be observed that the cost of perfect items paid by the buyer should be taken into account during the calculation of the effective cost of the items. If only the cost of the imperfect items sold is subtracted from the manufacturing cost, the effective cost of the items decreases as the percentage of the defective item increases which should not happen. Hence, the cost paid by the buyer for the perfect items is also subtracted from the manufacturing cost to calculate the effective cost of the items accurately. Let $\mathbb{E}[C_{ib}]$ be the effective cost of the items. Therefore, the effective cost of the items changes and it is given by

$$\begin{aligned}\mathbb{E}[C_{ib}] &= \{c_v nQ - c_b nQ - c_{bl} n(1-\mu)Q\}/n\mathbb{E}[T], \\ &= \frac{c_v D - c_b D - c_{bl}(1-\mu)D}{\mu}.\end{aligned} \tag{3.65}$$

The vendor's setup cost and holding cost of items at the vendor's location remains the same as the ones given by Equations (3.8) and (3.33), respectively.

Holding cost of imperfect items at the buyer's location:

As already discussed, the imperfect items of each delivery are held until the end of buyer's replenishment cycle and the holding cost is paid by the vendor. Let $\mathbb{E}[H_{vi}]$ be the expected holding cost of imperfect items at the buyer's location. Therefore, the vendor's average expected holding cost is given by

$$\begin{aligned}\mathbb{E}[H_{vi}] &= h_{vi}\frac{[(1-p_1)+(1-p_2)+...+(1-p_n)]\,Q\mathbb{E}[T]}{2n\mathbb{E}[T]}, \\ &= \frac{h_{vi}(1-\mu)Q}{2}.\end{aligned} \tag{3.66}$$

Vendor's expected total cost:

$$\begin{aligned}\mathbb{E}[TC(Q,n)]_v = & \frac{s_v D}{n\mu Q} + \frac{[c_v - c_b - c_{bl}(1-\mu)]D}{\mu} + h_v[\frac{Q(n-1)}{2} - \frac{QD(n-2)}{2\mu R}] \\ & + \frac{h_{vi}(1-\mu)Q}{2}\end{aligned} \tag{3.67}$$

3.3.3 Expected total cost of supply chain system

The expected total cost of the system is given by Equations (3.67), and (3.7). Therefore, the total cost of the supply chain is given by

$$\begin{aligned}\mathbb{E}[TC(Q,n)] = & \frac{s_v D}{n\mu Q} + \frac{s_b D}{\mu Q} + \frac{c_{bi} D}{\mu} + \frac{[c_v - c_b - c_{bl}(1-\mu)]D}{\mu} \\ & + h_{bp}\frac{Q(\mu^2+\delta^2)}{2\mu} + h_v[\frac{Q(n-1)}{2} - \frac{QD(n-2)}{2\mu R}] + \frac{h_{vi}Q(1-\mu)}{2}.\end{aligned} \tag{3.68}$$

3.3.4 Theoretical solutions

To calculate the optimal cost of the system, we partial differentiate the expected total cost of Equation (3.68) with respect to Q and n and equate it to zero and find the optimal Q and n and substitute them in the expected total cost function.

By differentiating Equation (3.68) with respect to Q, we get

$$\begin{aligned}\frac{\partial E[TC(Q,n)]}{\partial Q} = & \frac{-s_v D}{n\mu Q^2} - \frac{s_b D}{\mu Q^2} + \frac{h_v[n(\mu R - D) - (\mu R - 2D)]}{2\mu R} \\ & + \frac{h_{bp}(\mu^2+\delta^2)}{2\mu} + \frac{h_{vi}(1-\mu)}{2}.\end{aligned} \tag{3.69}$$

By equating the above Equation (3.69) to zero, we get

$$Q = \sqrt{\frac{2DR(s_v + ns_b)}{n\{h_v[n(\mu R - D) - (\mu R - 2D)] + h_{bp}R(\mu^2+\delta^2) + h_{vi}R\mu(1-\mu)\}}}. \tag{3.70}$$

By differentiating Equation (3.68) with respect to n, we get

$$\frac{\partial E[TC(Q,n)]}{\partial n} = \frac{-s_v D}{n^2\mu Q} + \frac{h_v Q(\mu R - D)}{\mu R}. \tag{3.71}$$

By equating above Equation (3.71) to zero, we get

$$n = \frac{1}{Q}\sqrt{\frac{2s_v DR}{h_v(\mu R - D)}}. \tag{3.72}$$

By substituting Equation (3.72) in Equation (3.70), we get

$$Q^2\left[h_{bp}R(\mu^2+\delta^2)\right] + Q^2\left[h_{vi}R\mu(1-\mu)\right] - Q^2[h_v(\mu R - 2D)]$$
$$+ Q[\sqrt{2h_v s_v DR(\mu R - D)}] = 2s_b DR + Q[\sqrt{2h_v s_v DR(\mu R - D)}].$$

$$Q = \sqrt{\frac{2s_b DR}{h_{bp}R(\mu^2+\delta^2) + h_{vi}R\mu(1-\mu) - h_v(\mu R - 2D)}}. \tag{3.73}$$

By substituting Equation (3.73) in Equation (3.72), we get

$$n = \sqrt{\frac{s_v[h_{bp}R(\mu^2+\delta^2) + h_{vi}R\mu(1-\mu)]}{s_b[h_v(\mu R - D)]}}. \tag{3.74}$$

3.3.5 Proof of convexity

To prove that the obtained values Q and n are optimal, we consider the Hessian matrix of second-order partial derivatives of the expected value of the total cost of the supply chain system.

Second-order partial derivative with respect to Q and n for the partial derivative of expected total cost with respect to Q.

$$\frac{\partial^2 E[TC(Q,n,N)]}{\partial^2 Q} = \frac{2s_v D}{n\mu Q^3} + \frac{2s_b D}{\mu Q^3}. \tag{3.75}$$

$$\frac{\partial^2 E[TC(Q,n,N)]}{\partial Q \partial n} = \frac{s_v D}{n^2\mu Q^2} + \frac{h_v}{2}(1 - \frac{D}{\mu R}). \tag{3.76}$$

Second-order partial derivative with respect to Q and n for the partial derivative of expected total cost with respect to n.

$$\frac{\partial^2 E[TC(Q,n,N)]}{\partial n \partial Q} = \frac{s_v D}{n^2\mu Q^2} + \frac{h_v}{2}(1 - \frac{D}{\mu R}). \tag{3.77}$$

$$\frac{\partial^2 E[TC(Q,n,N)]}{\partial^2 n} = \frac{2s_v D}{n^3 \mu Q}. \tag{3.78}$$

Therefore, the Hessian matrix H is given by

$$\begin{bmatrix} \frac{\partial^2 E[TC(Q,n,N)]}{\partial^2 Q} & \frac{\partial^2 E[TC(Q,n,N)]}{\partial Q \partial n} \\ \frac{\partial^2 E[TC(Q,n,N)]}{\partial n \partial Q} & \frac{\partial^2 E[TC(Q,n,N)]}{\partial^2 n} \end{bmatrix}$$

The elements in the first principal minor are all positive as given by Equations (3.75), and (3.78).

The elements in the second principal minor is given by D_1.

$$\begin{aligned} D_1 &= \left[\frac{\partial^2 E[TC(Q,n,N)]}{\partial^2 Q}\frac{\partial^2 E[TC(Q,n,N)]}{\partial^2 n}\right] \\ &- \left[\frac{\partial^2 E[TC(Q,n,N)]}{\partial n \partial Q}\frac{\partial^2 E[TC(Q,n,N)]}{\partial Q \partial n}\right], \\ &= \left(\frac{2s_v D}{n^3 \mu Q} + \frac{2s_b D}{\mu Q^3}\right)\left(\frac{2s_v D}{n^3 \mu Q}\right) - \left(\frac{s_v D}{n^2 \mu Q^2} + \frac{h_v}{2}\left(1 - \frac{D}{\mu R}\right)\right)^2. \end{aligned} \tag{3.79}$$

From Equation (3.72), we know that

$$n = \frac{1}{Q}\sqrt{\frac{2s_v DR}{h_v(\mu R - D)}}.$$

$$\frac{h_v}{2}\left(1 - \frac{D}{\mu R}\right) = \frac{s_v D}{n^2 \mu Q^2}. \tag{3.80}$$

Substituting Equation (3.80) in Equation (3.79), we get

$$\begin{aligned} D_1 &= \left(\frac{2s_b D}{\mu Q^3}\right)\left(\frac{2s_v D}{n^3 \mu Q}\right) + \left(\frac{4s_v^2 D^2}{n^4 \mu^2 Q^4}\right) - \left(\frac{2s_v D}{n^2 \mu Q^2}\right)^2, \\ &= \left(\frac{2s_b D}{\mu Q^3}\right)\left(\frac{2s_v D}{n^3 \mu Q}\right) + \left(\frac{4s_v^2 D^2}{n^4 \mu^2 Q^4}\right) - \left(\frac{4s_v^2 D^2}{n^4 \mu^2 Q^4}\right), \\ &= \left(\frac{2s_b D}{\mu Q^3}\right)\left(\frac{2s_v D}{n^3 \mu Q}\right). \end{aligned} \tag{3.81}$$

The second principal minor is also positive, proving that the Q and n are the minimum values for the expected total cost of the system.

3.4 Scrapping imperfect items at the vendor's location

In this case, the vendor sends quantity Q to the buyer, and the buyer inspects them. After inspection, the buyer holds the imperfect items for N cycles and transport them back to the vendor's location where the vendor disposes them. There is a fixed cost of d_{vi} for each salvage and variable cost of d_v for each item during salvage.

3.4.1 Buyer's expected total cost function

The buyer's cost function is the same as the one given in Equation (3.7).

3.4.2 Vendor's total expected cost function

In this case, the vendor's total cost includes the setup cost, manufacturing cost, holding cost of items at vendor's location, holding cost of imperfect items at the buyer's place, transport of imperfect items from the buyer's place to the vendor's place and salvage cost of imperfect items at the vendor's place.

The setup cost, manufacturing cost, holding cost of items at the vendor's location and the holding cost of imperfect items at the buyer's place is same as the ones given by Equations (3.8), (3.9), (3.10), and (3.33).

Salvage cost of imperfect items at the vendor's location:

The imperfect items are stored at the buyer's location for N cycles and then shipped to the vendor's location to salvage them at a fixed cost of d_{vi} for every salvage and a cost of d_v per each item. Therefore, the salvage cost of imperfect items at the vendor's location per year is given by

$$D_v = \mathbb{E}\{\sum_{i=1}^{N}(1-p_i)Qd_v + d_{vi}\}/N\mathbb{E}[T],$$
$$= \{(N - N\mu)Qd_v + d_{vi}\}D/N\mu Q,$$
$$= \frac{\{N(1-\mu)Qd_v + d_{vi}\}D}{N\mu Q}. \tag{3.82}$$

Transportation cost of imperfect items from the buyer's location to the vendor's location:

The imperfect items stored for N cycles must be transported to the vendor's location. There is a fixed cost of t_{vi} per each shipment and t_v per each item to be transported. Therefore, the total cost to transport imperfect items from the buyer's location to the vendor's location per year is given by

$$T_{vi} = \mathbb{E}\{t_{vi} + \sum_{i=1}^{N}(1-p_i)t_vQ\}/N\mathbb{E}[T],$$
$$= \{(N - N\mu)Qt_v + t_{vi}\}D/N\mu Q,$$
$$= \frac{\{N(1-\mu)Qt_v + t_{vi}\}D}{N\mu Q}. \tag{3.83}$$

Expected total cost of the vendor:

From the above Equations (3.8), (3.9), (3.10), (3.33), (3.82), and (3.83), the expected total cost of vendor is given by

$$\mathbb{E}[TC(Q,n,N)_v] = \frac{s_vD}{n\mu Q} + \frac{c_vD}{\mu} + h_v[\frac{Q(n-1)}{2} - \frac{QD(n-2)}{2\mu R}] + \frac{h_{vi}Q(1-\mu)(N+1)}{2}$$
$$+ \frac{\{N(1-\mu)Qt_v + t_{vi}\}D}{N\mu Q} + \frac{\{N(1-\mu)Qd_v + d_{vi}\}D}{N\mu Q}. \tag{3.84}$$

3.4.3 Expected total cost of the supply chain system

From Equations (3.7), and (3.84)

$$\begin{aligned}\mathbb{E}[TC(Q,n,N)] &= \frac{s_vD}{n\mu Q} + \frac{s_bD}{\mu Q} + \frac{c_vD}{\mu} + \frac{c_{bi}D}{\mu} + h_{bp}\frac{Q(\mu^2+\delta^2)}{2\mu} \\ &+ h_v[\frac{Q(n-1)}{2} - \frac{QD(n-2)}{2\mu R}] + \frac{h_{vi}Q(1-\mu)(N+1)}{2} \\ &+ \frac{\{N(1-\mu)Qt_v + t_{vi}\}D}{N\mu Q} + \frac{\{N(1-\mu)Qd_v + d_{vi}\}D}{N\mu Q}. \end{aligned} \tag{3.85}$$

3.4.4 Theoretical solutions

To calculate the optimal cost of the system, we partial differentiate the expected total cost Equation (3.85) with respect to Q, n and N, equate it to zero and find the optimal Q, n and N and substitute them in the expected total cost function.

By differentiating Equation (3.85) with respect to Q, we get

$$\begin{aligned}\frac{\partial E[TC(Q,n,N)]}{\partial Q} &= \frac{-s_vD}{n\mu Q^2} - \frac{s_bD}{\mu Q^2} + \frac{h_v[n(\mu R - D) - (\mu R - 2D)]}{2\mu R} + \frac{h_{bp}(\mu^2+\delta^2)}{2\mu} \\ &+ \frac{h_{vi}(1-\mu)(N+1)}{2} - \frac{D(t_{vi}+d_{vi})}{N\mu Q^2}. \end{aligned} \tag{3.86}$$

By equating the above Equation (3.86) to zero, we get

$$\begin{aligned}\frac{2DR[Ns_v + nNs_b + n(t_{vi}+d_{vi})])}{nNQ^2} &= h_v[n(\mu R - D) - (\mu R - 2D)] + h_{bp}R(\mu^2+\delta^2) \\ &+ h_{vi}R\mu(1-\mu)(N+1).\end{aligned}$$

$$Q = \sqrt{\frac{2DR[Ns_v + nNs_b + n(t_{vi}+d_{vi})]}{nN\left[h_v\{n(\mu R - D) - (\mu R - 2D)\} + h_{bp}R(\mu^2+\delta^2) + h_{vi}R\mu(1-\mu)(N+1)\right]}}. \tag{3.87}$$

By differentiating Equation (3.85) with respect to n, we get

$$\frac{\partial E[TC(Q,n,N)]}{\partial n} = \frac{-s_vD}{n^2\mu Q} + \frac{h_vQ(\mu R - D)}{\mu R}. \tag{3.88}$$

By equating the above Equation (3.88) to zero, we get

$$n = \frac{1}{Q}\sqrt{\frac{2s_v DR}{h_v(\mu R - D)}}. \tag{3.89}$$

By differentiating Equation (3.85) with respect to N, we get

$$\frac{\partial E[TC(Q,n,N)]}{\partial N} = \frac{-(t_{vi}+d_{vi})D}{N^2\mu Q} + \frac{h_{vi}Q(1-\mu)}{2}. \tag{3.90}$$

By equating the above Equation (3.90) to zero, we get

$$N = \frac{1}{Q}\sqrt{\frac{2(t_{vi}+d_{vi})D}{h_{vi}\mu(1-\mu)}}. \tag{3.91}$$

By substituting Equations (3.89), and (3.91) in Equation (3.87) and squaring it, we get

$$\begin{aligned}
Q^2\left[h_{bp}R(\mu^2+\delta^2)\right] &+ Q^2\left[h_{vi}R\mu(1-\mu)\right] - Q^2\left[h_v(\mu R - 2D)\right] \\
&+ Q(\sqrt{2DRh_v s_v(\mu R - D)}) + Q(R\sqrt{2D(t_{vi}+d_{vi})h_{vi}\mu(1-\mu)}) \\
&= 2DRs_b + Q\left[\sqrt{2DRh_v s_v(\mu R - D)}\right] \\
&Q\left[+R\sqrt{2D(t_{vi}+d_{vi})h_{vi}\mu(1-\mu)}\right].
\end{aligned}$$

$$Q = \sqrt{\frac{2s_b DR}{h_{bp}R(\mu^2+\delta^2) + h_{vi}R\mu(1-\mu) - h_v(\mu R - 2D)}}. \tag{3.92}$$

By substituting Equation (3.92) in Equations (3.89), and (3.91), we get

$$n = \sqrt{\frac{s_v[h_{bp}(\mu^2+\delta^2) + h_{vi}R\mu(1-\mu) - h_v(\mu R - 2D)]}{s_b h_v(\mu R - D)}}. \tag{3.93}$$

$$N = \sqrt{\frac{(t_{vi}+d_{vi})[h_{bp}(\mu^2+\delta^2) + h_{vi}R\mu(1-\mu) - h_v(\mu R - 2D)]}{s_b h_{vi}R\mu(1-\mu)}}. \tag{3.94}$$

3.4.5 Proof of convexity

To prove that the obtained values Q, n and N are optimal, we consider the Hessian matrix of second-order partial derivatives of the expected value of the total cost of the supply chain system.

Second-order partial derivative with respect to Q, n and N for the partial derivative of the expected total cost with respect to Q.

$$\frac{\partial^2 E[TC(Q,n,N)]}{\partial^2 Q} = \frac{2s_v D}{n\mu Q^3} + \frac{2s_b D}{\mu Q^3} + \frac{2D(d_{vi}+t_{vi})}{N\mu Q^3}. \tag{3.95}$$

$$\frac{\partial^2 E[TC(Q,n,N)]}{\partial Q \partial n} = \frac{s_v D}{n^2 \mu Q^2} + \frac{h_v}{2}(1 - \frac{D}{\mu R}). \tag{3.96}$$

$$\frac{\partial^2 E[TC(Q,n,N)]}{\partial Q \partial N} = \frac{(d_{vi}+t_{vi})D}{N^2 \mu Q^2} + \frac{h_{vi}}{2}(1-\mu). \tag{3.97}$$

Second-order partial derivative with respect to Q, n and N for the partial derivative of expected total cost with respect to n.

$$\frac{\partial^2 E[TC(Q,n,N)]}{\partial n \partial Q} = \frac{s_v D}{n^2 \mu Q^2} + \frac{h_v}{2}(1 - \frac{D}{\mu R}). \tag{3.98}$$

$$\frac{\partial^2 E[TC(Q,n,N)]}{\partial^2 n} = \frac{2s_v D}{n^3 \mu Q}. \tag{3.99}$$

$$\frac{\partial^2 E[TC(Q,n,N)]}{\partial n \partial N} = 0. \tag{3.100}$$

Second-order partial derivative with respect to Q, n and N for the partial derivative of expected total cost with respect to N.

$$\frac{\partial^2 E[TC(Q,n,N)]}{\partial N \partial Q} = \frac{(d_{vi}+t_{vi})D}{N^2 \mu Q^2} + \frac{h_{vi}}{2}(1-\mu). \tag{3.101}$$

$$\frac{\partial^2 E[TC(Q,n,N)]}{\partial N \partial n} = 0. \tag{3.102}$$

$$\frac{\partial^2 E[TC(Q,n,N)]}{\partial^2 N} = \frac{2D(d_{vi}+t_{vi})}{N^3 \mu Q}. \tag{3.103}$$

Similar to the case of scrapping imperfect items at the buyer's location, the hessian matrix is considered and convexity is proved.

The elements in the first principal minor are all positive as given by Equations (3.95), (3.99), and (3.103).

The elements in the second principal minor are D_1, D_2, and D_3.

$$
\begin{aligned}
D_1 &= (B_5B_9) - (B_6B_8), \\
&= \left(\frac{2s_vD}{n^3\mu Q}\frac{2D(d_{vi}+t_{vi})}{N^3\mu Q}\right) - 0, \\
&= \frac{4D^2s_v(d_{vi}+t_{vi})}{n^3N^3\mu^2Q^2}. \qquad (3.104)
\end{aligned}
$$

$$
\begin{aligned}
D_2 &= (B_1B_9) - (B_3B_7), \\
&= \left[\left(\frac{2s_vD}{n\mu Q^3} + \frac{2s_bD}{\mu Q^3} + \frac{2D(d_{vi}+t_{vi})}{N\mu Q^3}\right)\left(\frac{2D(d_{vi}+t_{vi})}{N^3\mu Q}\right)\right] \\
&- \left[\left(\frac{(d_{vi}+t_{vi})}{N^2\mu Q^2} + \frac{h_{vi}}{2}(1-\mu)\right)^2\right]. \qquad (3.105)
\end{aligned}
$$

From Equation (3.91), we know that

$$
N = \frac{1}{Q}\sqrt{\frac{2(t_{vi}+d_{vi})D}{h_{vi}\mu(1-\mu)}}.
$$

$$
\frac{h_{vi}}{2}(1-\mu) = \frac{(d_{vi}+t_{vi})D}{N^2\mu Q^2}. \qquad (3.106)
$$

Substituting Equation (3.106) in Equation (3.105), we get

$$
\begin{aligned}
D_2 &= \left[\left(\frac{2s_vD}{n\mu Q^3} + \frac{2s_bD}{\mu Q^3}\right)\left(\frac{2D(d_{vi}+t_{vi})}{N^3\mu Q}\right)\right] + \left[\left(\frac{2D(d_{vi}+t_{vi})}{N\mu Q^3}\right)\left(\frac{2D(d_{vi}+t_{vi})}{N^3\mu Q}\right)\right] \\
&- \left(\frac{D(d_{vi}+t_{vi})}{N^2\mu Q^2} + \frac{(d_{vi}+t_{vi})D}{N^2\mu Q^2}\right)^2, \\
&= \left[\left(\frac{2s_vD}{n\mu Q^3} + \frac{2s_bD}{\mu Q^3}\right)\left(\frac{2D(d_{vi}+t_{vi})}{N^3\mu Q}\right)\right] + \left(\frac{4(d_{vi}+t_{vi})^2D^2}{N^4\mu^2Q^4}\right) - \left(\frac{4(d_{vi}+t_{vi})^2D^2}{N^4\mu^2Q^4}\right), \\
&= \left[\left(\frac{2s_vD}{n\mu Q^3} + \frac{2s_bD}{\mu Q^3}\right)\left(\frac{2D(d_{vi}+t_{vi})}{N^3\mu Q}\right)\right]. \qquad (3.107)
\end{aligned}
$$

$$D_3 = (B_1B_5) - (B_2B_4),$$

$$= \left[\left(\frac{2s_vD}{n\mu Q^3} + \frac{2s_bD}{\mu Q^3} + \frac{2D(d_{vi}+t_{vi})}{N\mu Q^3}\right)\left(\frac{2s_vD}{n^3\mu Q}\right)\right] - \left[\left(\frac{s_vD}{n^2\mu Q^2} + \frac{h_v}{2}(1-\frac{D}{\mu R})\right)^2\right]. \tag{3.108}$$

From Equation (3.89), we know that

$$n = \frac{1}{Q}\sqrt{\frac{2s_vDR}{h_v(\mu R - D)}}.$$

$$\frac{h_v}{2}(1-\frac{D}{\mu R}) = \frac{s_vD}{n^2\mu Q^2}. \tag{3.109}$$

Substituting Equation (3.61) in Equation (3.108), we get

$$\begin{aligned}
D_3 &= \left[\left(\frac{2s_bD}{\mu Q^3} + \frac{2(d_{vi}+t_{vi})D}{N\mu Q^3}\right)\left(\frac{2s_vD}{n^3\mu Q}\right)\right] + \left[\left(\frac{2s_vD}{n\mu Q^3}\right)\left(\frac{2s_vD}{n^3\mu Q}\right)\right] \\
&- \left(\frac{s_vD}{n^2\mu Q^2} + \frac{s_vD}{n^2\mu Q^2}\right)^2, \\
&= \left[\left(\frac{2s_bD}{\mu Q^3} + \frac{2D(d_{vi}+t_{vi})}{N\mu Q^3}\right)\left(\frac{2s_vD}{n^3\mu Q}\right)\right] + \left(\frac{4s_v^2D^2}{n^4\mu^2Q^4}\right) - \left(\frac{4s_v^2D^2}{n^4\mu^2Q^4}\right). \\
&= \left[\left(\frac{2s_bD}{\mu Q^3} + \frac{2(d_{vi}+t_{vi})D}{N\mu Q^3}\right)\left(\frac{2s_vD}{n^3\mu Q}\right)\right].
\end{aligned} \tag{3.110}$$

All the elements in the second principal minor are also greater than zero.

The third principal minor is given by D_4.

$$D_4 = B_1\left[(B_5B_9) - (B_6B_8)\right] - B_2\left[(B_4B_9) - (B_6B_7)\right] - B_3\left[(B_4B_8) - (B_5B_7)\right]. \tag{3.111}$$

By substituting Equations (3.95), (3.96), (3.96), (3.98), (3.99), (3.100), (3.101), (3.102), and (3.103) in Equation (3.111) and solving it, we get

$$D_4 = \left(\frac{2s_vD}{n\mu Q^3} + \frac{2s_bD}{\mu Q^3}\right)\left(\frac{2s_vD}{n^3\mu Q}\frac{2(d_{vi}+t_{vi})D}{N^3\mu Q}\right) + \left(\frac{4s_v^2D^2}{n^4\mu Q^4}\frac{2(d_{vi}+t_{vi})D}{N^3\mu Q}\right). \tag{3.112}$$

The third principal minor is also a positive term. Since all the principal minors are positive, the values of Q, n and N have been proved to be optimal for the total expected cost of the supply chain system.

3.5 Discounting imperfect items at the vendor's location

Similar to the case as scrapping in the vendor's location, the buyer holds the imperfect items for N cycles as there is a transportation cost associated with it and then ships them to the vendor's location. They are then sold at the vendor's location at a discounted price.

3.5.1 Buyer's expected total cost function

Buyer's cost function is same as the one given in Equation (3.7).

3.5.2 Vendor's expected total cost function

The vendor's cost function involves setup cost, manufacturing cost, holding cost of items at the vendor's location, holding cost of imperfect items at the buyer's location and transportation cost of imperfect items from the buyer's location to the vendor's location. In this case, the buyer holds the imperfect items at the buyer's location for N cycles and then the vendor transports them to their location and sells them at a lower price c_{vl}. Similar to all the cases, the buyer pays for the holding cost of imperfect items at the buyer's location and to transport them to her location. The effective cost of the items is impacted by the fact that the imperfect items are stored for N cycles. Let $\mathbb{E}[C_{iv}]$ be the effective cost of items at the vendor's location. Therefore, vendor's expected effective cost of items is given by,

$$\begin{aligned}\mathbb{E}[C_{iv}] &= \frac{c_v nQ - c_b nQ}{n\mathbb{E}[T]} - \frac{c_{vl} N(1-\mu)Q}{N\mathbb{E}[T]},\\ &= \frac{c_v D - c_b D - c_{vl}(1-\mu)D}{\mu}.\end{aligned} \tag{3.113}$$

The vendor's setup cost, holding cost of items at the vendor's location, holding cost of imperfect items at the buyer's location and transportation cost of imperfect items from the buyer's location to the vendor's location is the same as the ones given by Equations (3.8), (3.33), (3.10), and (3.83).

Vendor's expected total cost:

From the above Equations (3.8), (3.113), (3.10), (3.33), and (3.83), the expected total cost of vendor is given by

$$\begin{aligned}\mathbb{E}[TC(Q,n,N)] &= \frac{s_v D}{n\mu Q} + \frac{c_v D - c_b D - c_{vl}(1-\mu)D}{\mu}\\ &+ h_v\{\frac{Q(n-1)}{2} - \frac{QD(n-2)}{2\mu R}\} + \frac{h_{vi}Q(1-\mu)(N+1)}{2}\\ &+ \frac{\{N(1-\mu)Qt_v + t_{vi}\}D}{N\mu Q}.\end{aligned} \tag{3.114}$$

3.5.3 Expected total cost of supply chain system

$$\begin{aligned}\mathbb{E}[TC(Q,n,N)] &= \frac{s_v D}{n\mu Q} + \frac{s_b D}{\mu Q} + \frac{c_v D - c_b D - c_{vl}(1-\mu)D}{\mu}\\ &+ h_v\{\frac{Q(n-1)}{2} - \frac{QD(n-2)}{2\mu R}\} + h_{bp}\frac{Q(\mu^2+\delta^2)}{2\mu}\\ &+ \frac{h_{vi}Q(1-\mu)(N+1)}{2} + \frac{\{N(1-\mu)Qt_v + t_{vi}\}D}{N\mu Q}.\end{aligned} \tag{3.115}$$

3.5.4 Theoretical solutions

To calculate the optimal cost of the system, we partial differentiate the expected total cost Equation (3.115) with respect to Q, n and N, equate it to zero and find the optimal

Q, n and N and substitute them in the expected total cost function.

By differentiating Equation (3.115) with respect to Q, we get

$$\frac{\partial E[TC(Q,n,N)]}{\partial Q} = \frac{-s_v D}{n\mu Q^2} - \frac{s_b D}{\mu Q^2} + \frac{h_v[n(\mu R - D) - (\mu R - 2D)]}{2\mu R} + \frac{h_{bp}(\mu^2+\delta^2)}{2\mu}$$
$$+ \frac{h_{vi}(1-\mu)(N+1)}{2} - \frac{Dt_{vi}}{N\mu Q^2}. \tag{3.116}$$

By equating the above Equation (3.116) to zero, we get

$$\frac{2DR[Ns_v + nNs_b + nt_{vi}])}{nNQ^2} = h_v[n(\mu R - D) - (\mu R - 2D)] + h_{bp}R(\mu^2+\delta^2)$$
$$+ h_{vi}R\mu(1-\mu)(N+1).$$

$$Q = \sqrt{\frac{2DR[Ns_v + nNs_b + nt_{vi}]}{nN\left[h_v\{n(\mu R - D) - (\mu R - 2D)\} + h_{bp}R(\mu^2+\delta^2) + h_{vi}R\mu(1-\mu)(N+1)\right]}}. \tag{3.117}$$

By differentiating Equation (3.115) with respect to n, we get

$$\frac{\partial E[TC(Q,n,N)]}{\partial n} = \frac{-s_v D}{n^2\mu Q} + \frac{h_v Q(\mu R - D)}{\mu R}. \tag{3.118}$$

By equating above Equation (3.71) to zero, we get

$$n = \frac{1}{Q}\sqrt{\frac{2s_v DR}{h_v(\mu R - D)}}. \tag{3.119}$$

By differentiating Equation (3.115) with respect to N, we get

$$\frac{\partial E[TC(Q,n,N)]}{\partial N} = \frac{-t_{vi}D}{N^2\mu Q} + \frac{h_{vi}Q(1-\mu)}{2}. \tag{3.120}$$

By equating the above Equation (3.120) to zero, we get

$$N = \frac{1}{Q}\sqrt{\frac{2t_{vi}D}{h_{vi}\mu(1-\mu)}}. \tag{3.121}$$

By substituting Equations (3.123), and (3.121) in Equation (3.117), we get

$$Q^2\left[h_{bp}R(\mu^2+\delta^2)\right] + Q^2\left[h_{vi}R\mu(1-\mu)\right] - Q^2\left[h_v(\mu R - 2D)\right]$$
$$+ Q\left[\sqrt{2DRh_v s_v(\mu R - D)} + R\sqrt{2Dt_{vi}h_{vi}\mu(1-\mu)}\right]$$
$$= 2DRs_b + Q\left[\sqrt{2DRh_v s_v(\mu R - D)} + P\sqrt{2Dt_{vi}h_{vi}\mu(1-\mu)}\right].$$

$$Q = \sqrt{\frac{2s_b DR}{h_{bp}R(\mu^2+\delta^2)+h_{vi}R\mu(1-\mu)-h_v(\mu R-2D)}}. \tag{3.122}$$

By substituting Equation (3.122) in Equations (3.123), and (3.121), we get

$$n = \sqrt{\frac{s_v[h_{bp}(\mu^2+\delta^2)+h_{vi}R\mu(1-\mu)-h_v(\mu R-2D)]}{s_b h_v(\mu R-D)}}. \tag{3.123}$$

$$N = \sqrt{\frac{t_{vi}[h_{bp}(\mu^2+\delta^2)+h_{vi}R\mu(1-\mu)-h_v(\mu R-2D)]}{s_b h_{vi}R\mu(1-\mu)}}. \tag{3.124}$$

3.5.5 Proof of convexity

To prove that the obtained values are Q, n and N, we consider the hessian matrix of second-order partial derivatives of expected value of total cost of the supply chain system.

Second-order partial derivative with respect to Q, n and N for the partial derivative of expected total cost with respect to Q.

$$\frac{\partial^2 E[TC(Q,n,N)]}{\partial^2 Q} = \frac{2s_v D}{n\mu Q^3}+\frac{2s_b D}{\mu Q^3}+\frac{2t_{vi}D}{N\mu Q^3}. \tag{3.125}$$

$$\frac{\partial^2 E[TC(Q,n,N)]}{\partial Q\partial n} = \frac{s_v D}{n^2\mu Q^2}+\frac{h_v}{2}(1-\frac{D}{\mu R}). \tag{3.126}$$

$$\frac{\partial^2 E[TC(Q,n,N)]}{\partial Q\partial N} = \frac{t_{vi} D}{N^2\mu Q^2}+\frac{h_{vi}}{2}(1-\mu). \tag{3.127}$$

Second-order partial derivative with respect to Q, n and N for the partial derivative of expected total cost with respect to n.

$$\frac{\partial^2 E[TC(Q,n,N)]}{\partial n\partial Q} = \frac{s_v D}{n^2\mu Q^2}+\frac{h_v}{2}(1-\frac{D}{\mu R}). \tag{3.128}$$

$$\frac{\partial^2 E[TC(Q,n,N)]}{\partial^2 n} = \frac{2s_v D}{n^3\mu Q}. \tag{3.129}$$

$$\frac{\partial^2 E[TC(Q,n,N)]}{\partial n\partial N} = 0. \tag{3.130}$$

Second-order partial derivative with respect to Q, n and N for the partial derivative of expected total cost with respect to N.

$$\frac{\partial^2 E[TC(Q,n,N)]}{\partial N\partial Q} = \frac{t_{vi} D}{N^2\mu Q^2}+\frac{h_{vi}}{2}(1-\mu). \tag{3.131}$$

$$\frac{\partial^2 E[TC(Q,n,N)]}{\partial N \partial n} = 0. \tag{3.132}$$

$$\frac{\partial^2 E[TC(Q,n,N)]}{\partial^2 N} = \frac{2t_{vi}D}{N^3 \mu Q}. \tag{3.133}$$

Similar to the case of scrapping imperfect items at the buyer's location, the hessian matrix is considered and convexity is proved.

The elements in the principal minor are all positive as given by Equations (3.125), (3.129), and (3.133) and are all positive.

The second principal minor elements is given by D_1, D_2 and D_3.

$$\begin{aligned} D_1 &= (B_5 B_9) - (B_6 B_8), \\ &= \left(\frac{2s_v D}{n^3 \mu Q} \frac{2t_{vi} D}{N^3 \mu Q} \right) - 0, \\ &= \frac{4D^2 s_v t_{vi}}{n^3 N^3 \mu^2 Q^2}. \end{aligned} \tag{3.134}$$

$$\begin{aligned} D_2 &= (B_1 B_9) - (B_3 B_7), \\ &= \left[\left(\frac{2s_v D}{n \mu Q^3} + \frac{2s_b D}{\mu Q^3} + \frac{2t_{vi} D}{N \mu Q^3} \right) \left(\frac{2t_{vi} D}{N^3 \mu Q} \right) \right] - \left[\left(\frac{t_{vi} D}{N^2 \mu Q^2} + \frac{h_{vi}}{2}(1-\mu) \right)^2 \right]. \end{aligned} \tag{3.135}$$

From Equation (3.121), we know that

$$N = \frac{1}{Q}\sqrt{\frac{2t_{vi}D}{h_{vi}\mu(1-\mu)}}.$$

$$\frac{h_{vi}}{2}(1-\mu) = \frac{t_{vi}D}{N^2 \mu Q^2}. \tag{3.136}$$

Substituting Equation (3.136) in Equation (3.135), we get

$$
\begin{aligned}
D_2 &= \left[\left(\frac{2s_vD}{n\mu Q^3}+\frac{2s_bD}{\mu Q^3}\right)\left(\frac{2t_{vi}D}{N^3\mu Q}\right)\right]+\left[\left(\frac{2t_{vi}D}{N\mu Q^3}\right)\left(\frac{2t_{vi}D}{N^3\mu Q}\right)\right] \\
&- \left(\frac{t_{vi}D}{N^2\mu Q^2}+\frac{t_{vi}D}{N^2\mu Q^2}\right)^2, \\
&= \left[\left(\frac{2s_vD}{n\mu Q^3}+\frac{2s_bD}{\mu Q^3}\right)\left(\frac{2t_{vi}D}{N^3\mu Q}\right)\right]+\left(\frac{4t_{vi}^2D^2}{N^4\mu^2Q^4}\right)-\left(\frac{4t_{vi}^2D^2}{N^4\mu^2Q^4}\right), \\
&= \left[\left(\frac{2s_vD}{n\mu Q^3}+\frac{2s_bD}{\mu Q^3}\right)\left(\frac{2t_{vi}D}{N^3\mu Q}\right)\right]. \qquad (3.137)
\end{aligned}
$$

$$
\begin{aligned}
D_3 &= (B_1B_5)-(B_2B_4), \\
&= \left[\left(\frac{2s_vD}{n\mu Q^3}+\frac{2s_bD}{\mu Q^3}+\frac{2t_{vi}D}{N\mu Q^3}\right)\left(\frac{2s_vD}{n^3\mu Q}\right)\right]-\left[\left(\frac{s_vD}{n^2\mu Q^2}+\frac{h_v}{2}(1-\frac{D}{\mu R})\right)^2\right]. \qquad (3.138)
\end{aligned}
$$

From Equation (3.119), we know that

$$
n = \frac{1}{Q}\sqrt{\frac{2s_vDR}{h_v(\mu R-D)}}.
$$

$$
\frac{h_v}{2}(1-\frac{D}{\mu R}) = \frac{s_vD}{n^2\mu Q^2}. \qquad (3.139)
$$

Substituting Equation (3.139) in Equation (3.138), we get

$$
\begin{aligned}
D_3 &= \left[\left(\frac{2s_bD}{\mu Q^3}+\frac{2t_{vi}D}{N\mu Q^3}\right)\left(\frac{2s_vD}{n^3\mu Q}\right)\right]+\left[\left(\frac{2s_vD}{n\mu Q^3}\right)\left(\frac{2s_vD}{n^3\mu Q}\right)\right]-\left(\frac{s_vD}{n^2\mu Q^2}+\frac{s_vD}{n^2\mu Q^2}\right)^2, \\
&= \left[\left(\frac{2s_bD}{\mu Q^3}+\frac{2t_{vi}D}{N\mu Q^3}\right)\left(\frac{2s_vD}{n^3\mu Q}\right)\right]+\left(\frac{4s_v^2D^2}{n^4\mu^2Q^4}\right)-\left(\frac{4s_v^2D^2}{n^4\mu^2Q^4}\right), \\
&= \left[\left(\frac{2s_bD}{\mu Q^3}+\frac{2t_{vi}D}{N\mu Q^3}\right)\left(\frac{2s_vD}{n^3\mu Q}\right)\right]. \qquad (3.140)
\end{aligned}
$$

All the elements in the second principal minor are also greater than zero.

The third principal minor is given by D_4.

$$
D_4 = B_1\left[(B_5B_9)-(B_6B_8)\right]-B_2\left[(B_4B_9)-(B_6B_7)\right]-B_3\left[(B_4B_8)-(B_5B_7)\right]. \qquad (3.141)
$$

By substituting Equations (3.125), (3.126), (3.126), (3.128), (3.129), (3.130), (3.131), (3.132), and (3.133) in Equation (3.141) and solving, we get

$$D_4 = \left(\frac{2s_v D}{n\mu Q^3} + \frac{2s_b D}{\mu Q^3}\right)\left(\frac{2s_v D}{n^3 \mu Q}\frac{2t_{vi} D}{N^3 \mu Q}\right) + \left(\frac{4s_v^2 D^2}{n^4 \mu Q^4}\frac{2t_{vi} D}{N^3 \mu Q}\right). \tag{3.142}$$

The third principal minor is also a positive term. Since all the principal minors are positive, the values of Q, n and N have been proved to be optimal for the total expected cost of the supply chain system.

3.6 Comparing the expected total costs of the system

It is assumed that the setup costs, holding costs, vendor's production cost, buyer's buying and inspection cost, buyer's demand and vendor's production rate, the average non-defective items and its deviation are constant in all the cases.

It is further denoted that scrapping imperfect items at the buyer's location is Case I, discounting imperfect items at the buyer's location as Case II, scrapping imperfect items at the vendor's location as Case III and discounting imperfect items at the vendor's location as Case IV.

3.6.1 Comparing Case I and Case II

If the total cost of Case I is greater than the total cost in Case II,

$$\frac{h_{vi}Q(1-\mu)}{2} - \frac{c_{bl}(1-\mu)D}{\mu} < \frac{h_{vi}Q(1-\mu)(N+1)}{2} + \frac{\{N(1-\mu)Qd_b + d_{bi}\}D}{N\mu Q},$$
$$-D(1-\mu)(d_b + c_{bl}) < \sqrt{2h_{vi}d_{bi}\mu(1-\mu)D}. \tag{3.143}$$

After comparison of both the total expected cost equations, as the above condition is always true and there is a negative term in the form of discounted price of imperfect items in Case II, the total expected cost in Case I is always greater than the total expected cost in Case II.

3.6.2 Comparing Case I and Case III

If the total cost of Case I is greater than the total cost of Case III,

$$\frac{D(1-\mu)(t_v+d_v)}{\mu} + \sqrt{\frac{2h_{vi}D(1-\mu)(t_{vi}+d_{vi})}{\mu}} > \frac{D(1-\mu)d_b}{\mu} + \sqrt{\frac{2h_{vi}D(1-\mu)d_{bi}}{\mu}},$$
$$\frac{(t_v+d_v)-d_b}{\sqrt{t_{bi}}-\sqrt{t_{vi}+d_{vi}}} > \sqrt{\frac{2h_{vi}\mu}{D(1-\mu)}}. \tag{3.144}$$

From Equation (3.144), if the numerator on the left hand side is positive, the denominator is negative. If the numerator is negative on the left hand side, then the denominator is a positive term. Therefore, the total cost in Case I is always less than the total cost in Case III.

3.6.3 Comparing Case I and Case IV

If the total cost of Case IV is greater than the total cost of Case I

$$\sqrt{\frac{2h_{vi}t_{vi}D(1-\mu)}{\mu}} + \frac{D(1-\mu)(t_v-c_{vl})}{\mu} > \sqrt{\frac{2h_{vi}d_{bi}D(1-\mu)}{\mu}} + \frac{D(1-\mu)d_b}{\mu},$$
$$\frac{t_v-c_{vl}-d_b}{\sqrt{d_{bi}}-\sqrt{t_{vi}}} > \sqrt{\frac{2\mu h_{vi}}{D(1-\mu)}}. \tag{3.145}$$

From Equation (3.145), on the left hand side, if the term $t_v - d_b$ is negative, then the term $\sqrt{d_{bi}} - \sqrt{t_{vi}}$ is positive. If the term $t_v - d_b$ is positive, then the term $\sqrt{d_{bi}} - \sqrt{t_{vi}}$ is negative. Hence, the total expected cost in Case IV is also less than the total cost in Case I.

3.6.4 Comparing Case II with Case III

If the total cost of Case III is greater than the total cost in Case II,

$$\begin{aligned}\frac{h_{vi}Q(1-\mu)}{2} - \frac{c_{bl}(1-\mu)D}{\mu} &< \frac{h_{vi}Q(1-\mu)(N+1)}{2} \\ &\quad + \frac{\{N(1-\mu)Q(d_v+t_v)+(d_{bi}+t_{vi})\}D}{N\mu Q}, \\ -D(1-\mu)(d_v+t_v+c_{bl}) &< \sqrt{2h_{vi}(d_{bi}+t_{vi})\mu(1-\mu)D}. \end{aligned} \tag{3.146}$$

After comparison of both the total expected cost equations, as the above condition is always true and there is a negative term in the form of discounted price of imperfect items in Case II, the total expected cost in Case III is always greater than the total expected cost in Case II.

3.6.5 Comparing Case II with Case IV

If the total cost of Case IV is greater than the total cost in Case II,

$$\begin{aligned}\frac{h_{vi}Q(1-\mu)}{2} - \frac{c_{bl}(1-\mu)D}{\mu} &< -\frac{c_{vl}(1-\mu)D}{\mu} + \frac{h_{vi}Q(1-\mu)(N+1)}{2} \\ &\quad + \frac{\{N(1-\mu)Qt_v + +t_{vi}\}D}{N\mu Q}, \\ -D(1-\mu)(t_v+c_{bl}-c_{vl}) &< \sqrt{2h_{vi}t_{vi}\mu(1-\mu)D}. \end{aligned} \tag{3.147}$$

From Equation (3.147), the term on the left side is always negative. Therefore, the total cost of Case II is always lesser than the total cost of Case IV.

3.6.6 Comparing Case III and Case IV

If the total cost of Case III is greater than the total cost of Case IV, we get

$$\begin{aligned}\sqrt{\frac{2(t_{vi}+d_{vi})h_{vi}(1-\mu)D}{\mu}}+\frac{(1-\mu)(t_v+d_v)D}{\mu} &> \sqrt{\frac{2t_{vi}h_{vi}(1-\mu)D}{\mu}} \\ &+ \frac{(1-\mu)D(t_v-c_{vl})}{\mu}, \\ \frac{D(1-\mu)(d_v+c_{vl})}{\mu} &> \sqrt{\frac{2Dh_{vi}(1-\mu)}{\mu}}(\sqrt{t_{vi}}-\sqrt{t_{vi}+d_{vi}}).\end{aligned} \tag{3.148}$$

For any given positive value of t_{vi}, the value of $(\sqrt{t_{vi}}-\sqrt{t_{vi}+d_{vi}})$ is going to be negative. Therefore, the above Expression (3.148) is

$$\frac{(d_v+c_{vl})}{(\sqrt{t_{vi}}-\sqrt{t_{vi}+d_{vi}})} < \sqrt{\frac{2\mu h_{vi}}{(1-\mu)D}}. \tag{3.149}$$

Since the above condition is true for any given value of t_{vi}, the total cost of discounting imperfect items at the vendor's location is always lesser than the total cost of scrapping imperfect items at the vendor's location.

From all the above comparisons, as the total cost of Case II is always less than the cost of all other cases, the total cost of the system is always the least in the Case II. As the total cost in Case IV is less than the total cost in Case I and Case III, the total cost of the system is second least in Case IV. Also, the total cost in Case III is greater than the total cost in Case I, total cost in Case III is always the greatest. Therefore, the total cost in the increasing order is Case II < Case IV < Case I < Case III.

Chapter 4

Example and Sensitivity Analysis

In this chapter, a numerical example is provided to show how the models work and sensitivity analysis is performed to analyze the behaviour of the four systems .

4.1 An example

Consider an example where the parameters are as follows. $D = 5000, R = 23000, s_b = 130, s_v = 2200, h_{bp} = 12, h_v = 8, h_{vi} = 3.5, \mu = 0.9, \delta = 0.03, c_v = 2, c_{bi} = 0.2, c_b = 2.4, c_{vl} = 1.9, c_{bl} = 1.8, t_v = 1, t_{vi} = 15, d_v = 0.6, d_{vi} = 25, d_b = 0.8, d_{bi} = 30$.

The result is obtained by calculating the value of Q, then n and N. If Q is not an integer, it is rounded to the next closest integer. If n and N are not integer values, the floor and roof values of n and N are considered and are substituted in the total cost equation in combinations to find the least total expected cost of the system.

By solving in the above procedure, the obtained values for Case I are $Q = 456, n = 4, N = 2, \mathbb{E}[TC(Q, n, N)] = 15287$. In Case II, as imperfect items are not stored for N cycles before discounting them, we only calculate the values of Q and n. Following the same procedure, the obtained values are $Q = 456, n = 4, \mathbb{E}[TC(Q, n)] = 13591$. In Case III, the obtained values are $Q = 456, n = 4, N = 3, \mathbb{E}[TC(Q, n, N)] = 15780$. In Case IV, the obtained values are $Q = 456, n = 4, N = 2, \mathbb{E}[TC(Q, n, N)] = 14251$.

In this example, the expected total cost of the system is least when the imperfect items are discounted at the vendor's place and is the highest in the case of scrapping imperfect items at the vendor's place. However, as per the conditions provided in the previous chapter, the total expected cost varies. Therefore, given the conditions, the total expected cost of the system is the least in the case of discounting imperfect items at the buyer's place as the imperfect items are not stored for N cycles and also because the imperfect items are sold at a discounted price.

4.2 Sensitivity analysis

The input parameters such as $\mu, D/R, h_v/h_{vi}$ and h_{bp}/h_v are varied to see the behaviour of the system

The percentage of perfect items in each shipment is varied to see the system behaviour in the four cases. The change in the quantity per each shipment and the number of shipments is given by Table 4.1. There is no column for N for Case II, as defective items are not stored for N cycles in Case II.

Table 4.1: Comparing different percentages of perfect items and subsequent change in the buyer's order quantity

μ	Q	n	N(Case I)	N(Case III)	N(Case IV)
0.95	436	4	3	4	2
0.9	456	4	2	3	2
0.85	476	4	2	2	1
0.8	498	4	2	2	1
0.75	521	4	1	2	1

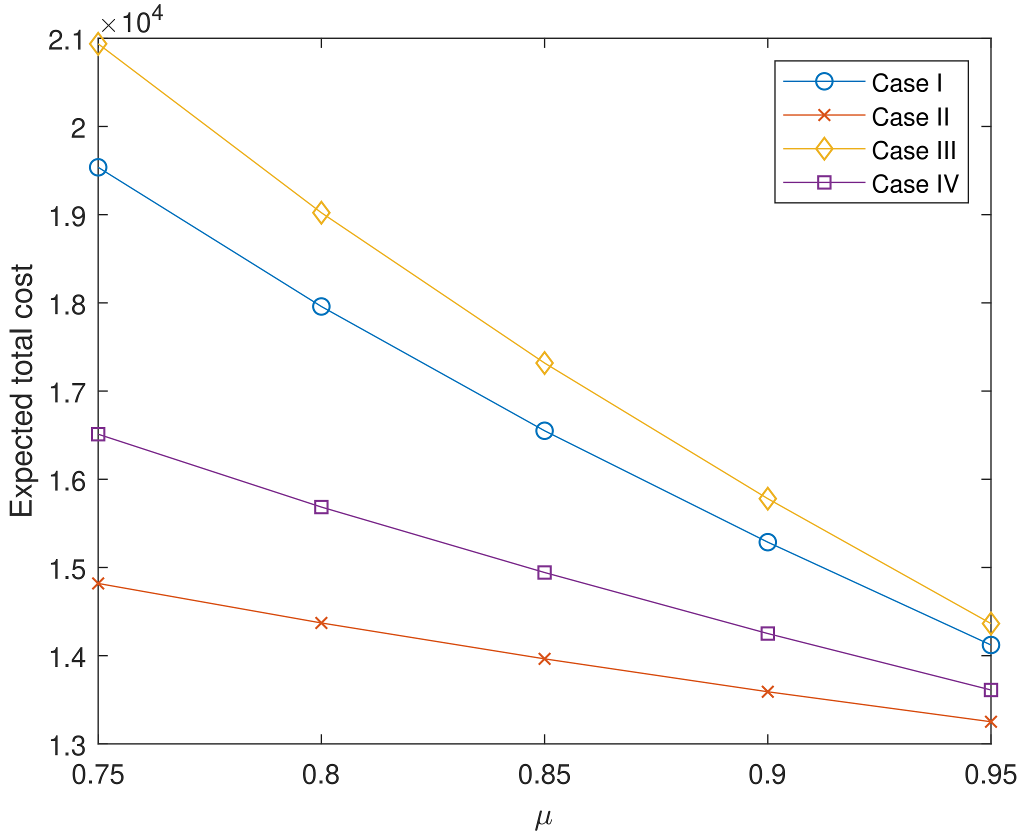

Figure 4.1: Percentage of perfect items vs expected total cost

From Figure 4.1, it is seen that the total cost and quantity per each shipment in all four cases increases as the percentage of perfect items decreases. The vendor has to produce more items in order to satisfy the demand of the buyer as the percentage of perfect items decreases and this leads to an increase in the expected total cost of the system. It is also seen that the total cost is least in any given condition for the case where the imperfect items are discounted at the buyer's location.

In the next analysis, the ratio of demand to production rate is varied, and the system behaviour is given by Table 4.2.

Table 4.2: Changing ratio of buyer's demand to vendor's production rate and subsequent change in quantity per each shipment

D/R	Q	n	N(Case I)	N(Case III)	N(Case IV)
0.2	456	4	2	3	2
0.3	400	6	3	3	2
0.4	371	6	3	3	2
0.5	347	8	3	4	2
0.6	324	9	3	4	2

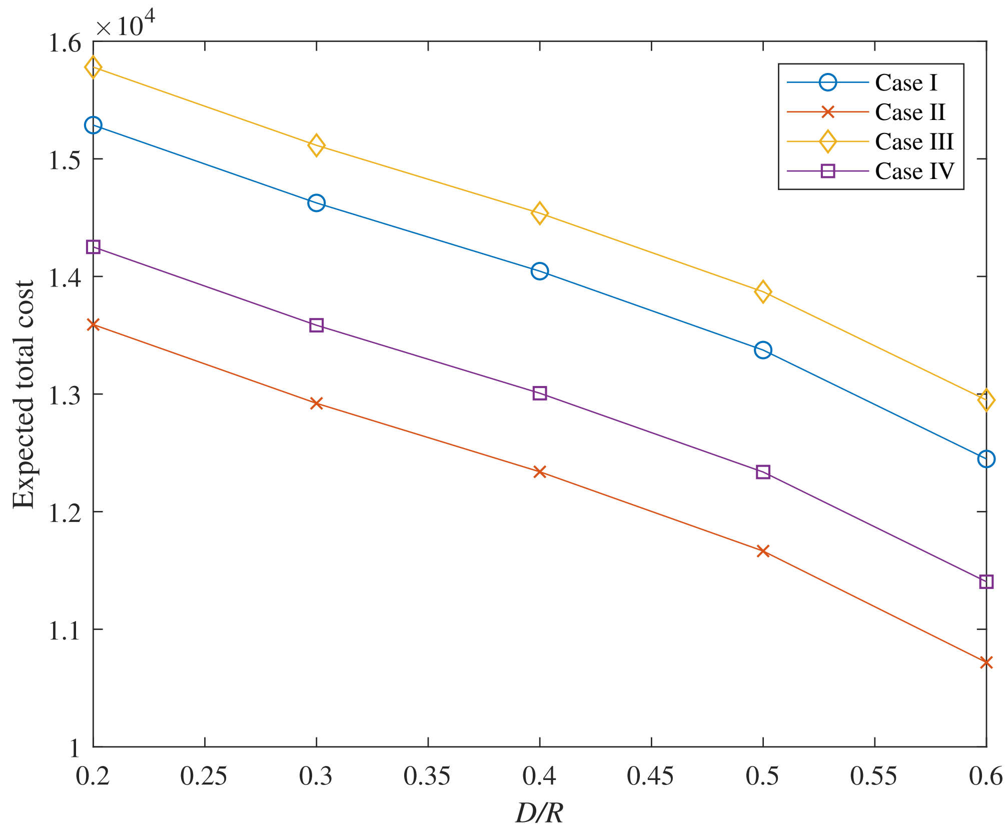

Figure 4.2: Ratio of demand to production rate vs expected total cost

From Figure 4.2, as the ratio increases, the quantity per each shipment and total expected cost of the system decreases. As the production rate decreases, the holding cost at the vendor's place decreases, and in turn, the total cost of the system goes down. Also, as the vendor produces fewer items in each cycle as the production rate goes down, the quantity for each shipment decreases and the number of shipments increases. As, the production rate goes down,as the number of items in each shipment decreases, the number of defective items in each shipment also decreases and so the buyer holds the imperfect items for more number of cycles.

In the next analysis, the ratio of the holding cost of perfect items at the buyer's location to holding cost of imperfect items at the buyer's location is varied and the behaviour of the system under this condition is given by Table 4.3.

Table 4.3: Changing h_{bp}/h_v and subsequent change in the buyer's order quantity

h_{bp}/h_v	Q	n	N(Case I)	N(Case III)	N(Case IV)
2	425	5	3	3	2
3	400	7	3	3	2
4	388	8	3	3	2
5	382	10	3	3	2
6	379	11	3	3	2

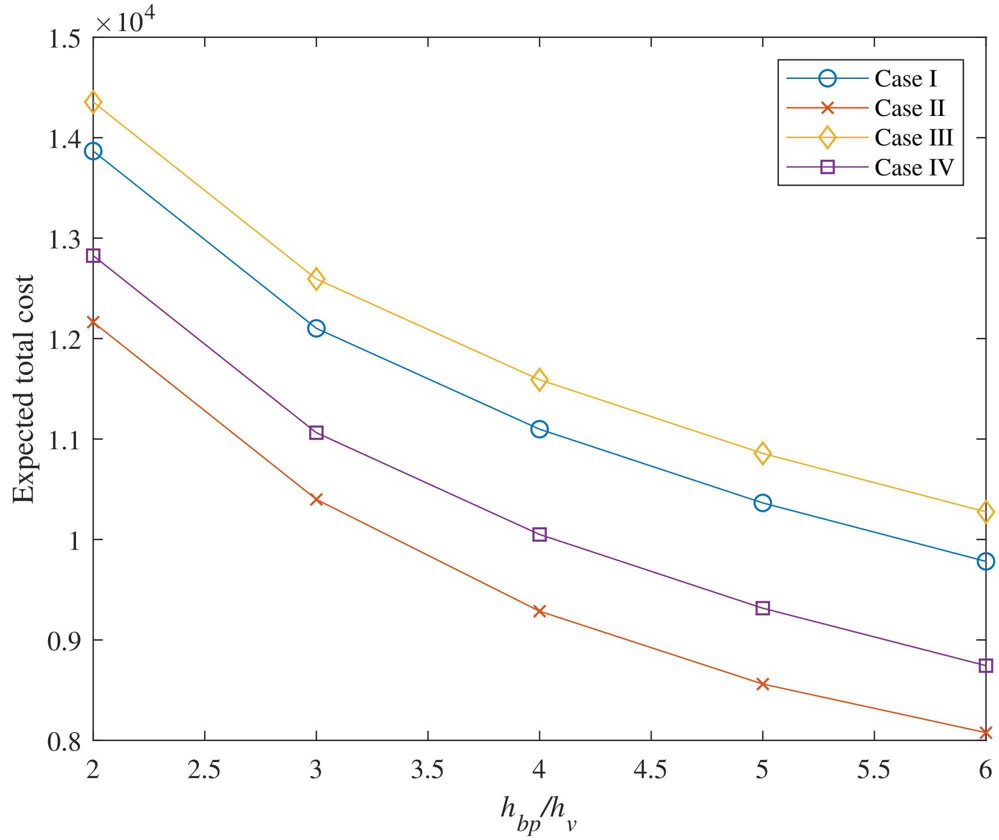

Figure 4.3: h_{bp}/h_v vs expected total cost

From Figure 4.3, as the ratio increases, the holding cost at the vendor's place decreases and the total expected cost of the system goes down. From Table 4.3, we also see that at a higher holding cost at the vendor's place, the vendor ships more quantity with less number of shipments. But, as the vendor's holding cost goes down, the vendor ships a lesser quantity with more number of shipments. And also, as the quantity decreases, the buyer holds the imperfect items for more number of cycles.

In the next analysis, h_v/h_{vi} is changed, the behaviour of the system is studied, and it is presented in Table 4.4.

From Figure 4.4 and Table 4.4, as the ratio decreases, or in other words, as the holding cost of imperfect items at the buyer's location increases, the total cost of the system increases. Also, as the holding cost of imperfect items increases, the quantity does not

Table 4.4: Changing h_v/h_{vi} and subsequent change in the buyer's order quantity

h_v/h_{vi}	Q	n	N(Case I)	N(Case III)	N(Case IV)
10	463	4	4	5	3
8	462	4	4	4	3
6	461	4	3	4	2
4	459	4	3	3	2
2	454	4	2	3	2

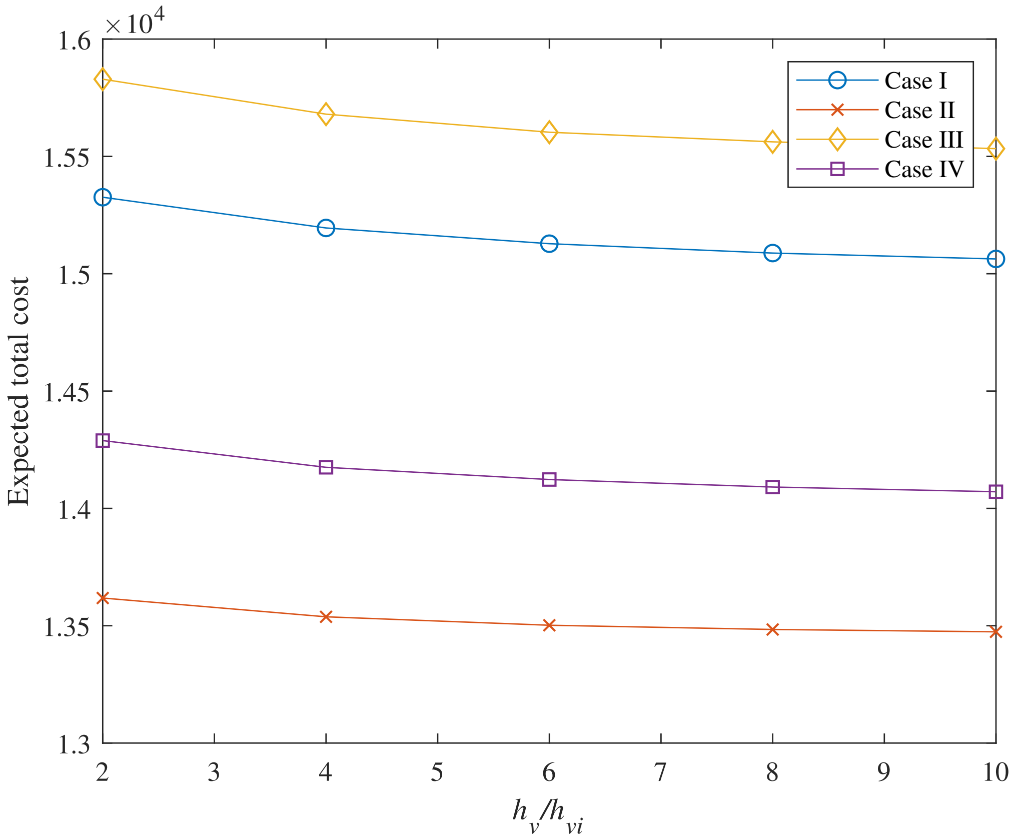

Figure 4.4: h_v/h_{vi} vs expected total cost

significantly change, but the number of cycles and imperfect items held decrease.

Chapter 5

Conclusion

In this project, four models were developed using the renewal-reward theorem for a single-vendor and single-buyer inventory system to optimize the total expected cost. Two models were regarding scrapping imperfect items at buyer's and vendor's locations. The other two models were regarding discounting the imperfect items at buyer's and vendor's locations. Calculating the expected cycle time and the expected total cost of the system by considering the disposal methods of imperfect items was the major outcome of this project. Comparisons were made between these models and sufficient conditions were provided to adopt a particular model. The holding cost of imperfect items highly influences the total cost of the system and this is seen from the theoretical and numerical results. Another outcome is that given any input parameters, the expected total cost of the system is always the least for the case where the buyer sells the imperfect items at a discounted price. The second least cost is observed in the case where discounting defective items at the vendor's location is done. The total cost of the system is the highest when the vendor decides to scrap the defective items at the vendor's location. It is because the buyer holds the imperfect items for one cycle only, and also the imperfect items are sold at a discounted price. Based on the theoretical and numerical results, it was observed that the total expected cost of the system increases as the percentage of imperfect items increases in each shipment for all the models.

Reworking of imperfect items to meet the demand of the buyer is an interesting topic to work. In this case, the items are stored at the buyer's location for certain number of replenishment cycles and then returned to the vendor to rework them to make them perfect items that are used to satisfy the buyer's demand. It is also interesting to extend the work if multiple items were being delivered by the vendor to the buyer.

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