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Abstract

Primbs *et al.* (2007) proposed an option pricing method using a pentanomial lattice that incorporates mean, volatility, skewness, and kurtosis. This approach is very useful when the return of the underlying asset follows a lognormal distribution. However, Primbs *et al.* (2007) claimed that "with four moments, one could conceivably use a quadrinomial lattice (i.e., four branches); however, the recombination conditions along with the requirement of non-negative probabilities are quite limiting in terms of the range of skewness and kurtosis that can be captured". In this research, as a refutation, a quadrinomial lattice model has been developed incorporating mean, volatility, skewness, and kurtosis; and it has been shown that the conditions for the non-negative probabilities are the same as the conditions obtained for the pentanomial lattice in Primbs *et al.* (2007). Several numerical examples are presented to compare the result obtained from the quadrinomial lattice with that of the pentanomial lattice.

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List of Abbreviations

Π	Value of the portfolio
ϕ	Continuous compounded return of the underlying stochastic variable when it is going along the upward branch
λ	Scaler
α	Parameter or jump size
ϑ	Expected rate of return per year
ϱ	Mean of $ln(S_T)$
ω	Standard deviation of $ln(S_T)$
μ_i	i^{th} central moment
σ	Volatility of the stock price per year
τ	Time interval between two steps
c_i	i^{th}cumulant
Ψ	Value of the Call option
f	Option price
k	Daily kurtosis
$\hat{k}$	Yearly kurtosis
Y	Strike price of an underlying asset
L	Number of branches
m_i	i^{th} raw moment
N	Number of steps or days
p	Branch probability
Ω	Value of the Put option
q	Probability in terms of branch probability
r_f	Risk-free interest rate
ς	Daily skewness
$\hat{\varsigma}$	Yearly skewness

S	Spot price of an underlying asset
t	Time
T	Maturity time or expiration date
X	Variable that follows Levy process
ϵ	Random variable that follows Wiener process
Z	Discrete random variable

CHAPTER 1

1.1 Introduction

In the last 25 years, derivatives such as options became extremely important in the world of finance. An option is a contract between a buyer and a seller that gives the buyer a right, not an obligation, to trade an underlying asset on or before an expiration date, at an agreed price (Hull, 2006). In return for granting the option, the buyer pays an amount of money to the seller that is called the option premium or option price. On the basis of when the option is exercised, options can be classified as: European options and American options. European options can be exercised only on the expiration date, while American options can be exercised at any time until the expiration date. The option can also be categorized in two classes on the basis of whether to buy or sell an underlying asset. One is the call option that offers the buyer the right to buy the asset. The other is the put option that gives the buyer the right to sell the asset.

In the option price theory, a breakthrough was made in 1973, when Black and Scholes (1973) presented the first satisfactory equilibrium option pricing model based on the risk neutral arbitrage argument. This model is well known as the Black-Scholes Model. Robert Merton extended their model for many subsequent studies (e.g., Merton, 1973). Unfortunately, the Black-Scholes Model can only be used to value simple options such as European options; it can not be used to value more sophisticated options such as American options. Moreover, as the model provides a closed-form solution to a partial differential equation, it limits its application when a partial differential equation can not be derived for a particular option contract. Therefore, different alternative techniques have been proposed for pricing the options.

Among many option pricing techniques, one well-known method is the lattice approach. Cox *et al.* (1979) have developed a binomial lattice approach using the fundamental economic principles of option valuation which is no arbitrage arguments. The binomial lattice approach, also known as CRR model, can be used to value a wide range of options when the return of the underlying asset follows a normal distribution. The model has been developed by matching the two moments (mean and variance) of a discrete random variable over small time interval with those of a continuous random variable. After this flourishing attempt, many multinomial lattice methods have been proposed that can be used to value more complex options on several underlying variables. Boyle (1988) has extended the CRR binomial lattice to a trinomial lattice for a single underlying variable. Using the results of the trinomial lattice, Boyle (1988) has created a pentanomial lattice that can handle the situation where the payoff from the option depends on two underlying variables. By matching the characteristic function of a discrete distribution with that of a continuous lognormal distribution, Boyle *et al.* (1989) have developed a lattice model for multivariate contingent claims. Kamrad and Ritchken (1991) have proposed an alternative way of valuing the contingent claims on one or more underlying variables. This has been accomplished by matching the first two moments of a discrete distribution with those of a continuous normal distribution. Kamrad and Ritchken (1991) have also showed that the convergence rate of the trinomial lattice is much higher than that of the binomial lattice. Moreover, for two state underlying variables, the pentanomial lattice model possesses a higher convergence rate than the quadrinomial lattice.

All the models stated above have matched only the first two moments, mean and variance, as the normal distribution was considered for the return on the stock. However, since the return on the stock is assumed to have the lognormal distribution, the third and fourth moments (skewness and kurtosis) must be included. Rubinstein (1994) has proposed a lattice model that incorporates skewness and kurtosis by using an Edgeworth expansion. Yamada *et al.* (2004) and Primbs *et al.* (2007) have explored the issue of incorporating skewness and kurtosis directly in a pentanomial lattice model using a moment-matching procedure. Recombination conditions along with the requirement of non-negative probabilities were also obtained in terms of skewness and kurtosis. However, Primbs *et al.* (2007) have claimed that one could possibly use four moments by developing a quadrinomial lattice (four branches), but the requirement of positive probabilities is relatively limiting for the quadrinomial lattice in terms of the range of skewness and kurtosis that can be captured.

In this research, a quadrinomial lattice that incorporates skewness and kurtosis has been developed, and conditions for the non-negative probabilities have been derived. The conditions obtained for the quadrinomial lattice are same as the conditions obtained for the pentanomial lattice in Primbs *et al.* (2007).

The report is organised as follows: Chapter 2 provides a brief introduction of the option, types of option and basic methodologies to calculate the option prices. Basic models for the option pricing include the well known Black-Scholes model (Black *et al.*, 1973), Monte Carlo simulation techniques (e.g., Boyle, 1977), finite difference methods (e.g., Brennan and Schwartz, 1978) and the binomial lattice approach (Cox *et al.*, 1979). In addition, brief summaries of the trinomial lattice model (Boyle, 1988), the pentanomial lattice for two underlying variables (Boyle, 1988), the quadrinomial lattice model for two underlying variables (Boyle *et al.*, 1989) and the pentanomial lattice model for two underlying variables (Kamrad and Ritchken, 1991) have been presented. It also reviews the pentanomial lattice model developed in Primbs *et al.* (2007) for a single underlying variable that includes skewness and kurtosis along with the mean and volatility. Chapter 3 presents the quadrinomial lattice model development procedure. Furthermore, the conditions for positive branch probabilities have also been found and shown that these conditions are same as the conditions obtained for the pentanomial lattice in Primbs *et al.* (2007). Chapter 4 provides numerical examples considering a European call option. This chapter shows how to calculate the option prices using the quadrinomial lattice model developed in Chapter 3. The results obtained from the quadrinomial lattice model (developed in Chapter 3) and the pentanomial lattice model (proposed in Primbs *et al.* (2007)) are compared based on convergence, effect of volatility, and volatility smiles and smirks for different maturity periods. Chapter 5 concludes the results.

CHAPTER 2

This chapter provides a brief introduction to the basic concepts of the option and some models and methods for the option pricing. An option is an agreement between a buyer and a seller that gives its buyer a right, but not an obligation, to buy or sell an underlying asset at a specified price on or before a specified date. In return for granting the option, the seller collects an amount of money from the buyer that is called the option premium or option price. There are different techniques available for option pricing. The most commonly used techniques are the Black-Scholes model, Monte Carlo simulation technique, finite difference methods and the lattice approaches.

2.1 Literature Review more about options

An option is a financial instrument that offers a right to the holder, but not a responsibility, to be engaged in a future transaction of the underlying asset at a specified price at any time on or before a given date (Hull, 2006). The specified price is also called the strike price and the given date is called the expiration date or maturity date. Based on when an option can be exercised, there are two types of options: European options and American options. European options can be exercised only on the expiration date, whereas American options can be exercised at any time until the maturity date. Options can also be categorized based on the right to buy or sell an asset, such as a call option and a put option. A call option gives the buyer the right to buy an asset, while a put option offers the buyer the right to sell an asset. When the spot price (price of the underlying asset) exceeds the strike price, a call option should be exercised. A put option should be exercised when the spot price is less than the strike price. Let the strike price be Υ, and the spot price of the underlying asset at the date of exercise be S. If Ψ is the value of a call option and Ω is the value of a put option, then they can be expressed as

$$\Psi = \max(S - \Upsilon, 0) \quad \text{and} \quad \Omega = \max(\Upsilon - S, 0). \tag{2.1}$$

Since different option pricing models are available, we will briefly discuss them as follows.

2.1.1 Black-Scholes model

In finance history, option pricing theory underwent a revolutionary change in 1973, when Black and Scholes (1973) presented the first option pricing model. They have shown how the no arbitrage argument can be used to derive a partial differential equation to describe the relations between the value of an option and the price of its underlying asset. The concept of no arbitrage is the most significant tool for the analysis of derivatives. In finance, an arbitrage is an opportunity to take the advantage from the price differential between two markets (Hull 2002). If there is any arbitrage opportunity, one can make profit by simultaneously buying and selling the same assets in two different markets. However, the Black-Scholes model can only be used for European options, since the model is developed under the assumption that option can be exercised only at the expiration.

The Black-Scholes model considered that the stock price, S, follows the geometric Brownian motion, which is given by

$$dS = \vartheta S dt + \sigma S d\epsilon, \tag{2.2}$$

where ϑ is the expected return on the stock price and σ is the volatility, t is the time and ϵ is a variable that follows the Wiener process.

Now from Itô's lemma (Hull, 2006), the process followed by any function G of S and t is given by

$$dG = \left(\frac{\partial G}{\partial S}\vartheta S + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial G}{\partial S}\sigma S d\epsilon. \tag{2.3}$$

Now if f is the price of the option contingent on S, then f must be some function of S and t. Therefore, from Equation (2.3),

$$df = \left(\frac{\partial f}{\partial S}\vartheta S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial f}{\partial S}\sigma S d\epsilon. \tag{2.4}$$

The Wiener process can be eliminated by choosing an appropriate portfolio having a short position of one derivative and a long position of $\frac{\partial f}{\partial S}$ number of shares.

Define Π as the value of portfolio then,

$$\Pi = -f + \frac{\partial f}{\partial S}S \tag{2.5}$$

and the change in the value of the portfolio, $\delta\Pi$, for short time interval, δt, is given by

$$\delta\Pi = -\delta f + \frac{\partial f}{\partial S}\delta S. \tag{2.6}$$

By substituting Equations (2.2) and (2.4) in Equation (2.6), we get,

$$\delta\Pi = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\delta t. \tag{2.7}$$

Equation (2.7) does not have the stochastic term $d\epsilon$; therefore, the portfolio can be said to be riskless during time period δt under no arbitrage argument. Therefore,

$$\delta\Pi = r_f \Pi \delta t, \tag{2.8}$$

where r_f is the risk-free interest rate. Then the following partial differential equation, known as the Black-Scholes differential equation, can be obtained by substituting Equations (2.5) and (2.7) into Equation (2.8).

$$\frac{\partial f}{\partial t} + r_f S \frac{\partial f}{\partial S} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 = r_f f. \qquad (2.9)$$

Black and Scholes (1973) found a closed form solution for the European call option using the boundary condition

$$\Psi = \max(S - \Upsilon, 0) \quad \text{when } t = T$$

and for the European put option using the boundary condition

$$\Omega = \max(\Upsilon - S, 0) \quad \text{when } t = T.$$

A large number of option prices can be calculated in a simple manner using the Black-Scholes model. However, since the Black-Scholes model is only applicable to European option pricing, other numerical methods such as the Monte Carlo simulation technique, finite difference methods, and lattice approaches have been proposed as alternate option pricing methods for both European options and American options.

2.1.2 Monte Carlo simulation technique

A Monte Carlo simulation method is a computational algorithm that generates a number of sample paths to value an option. In an option pricing procedure, numbers of random sample paths are generated to obtain the expected payoff of an option in a risk-neutral world. The expected payoff is then discounted back at a risk-free rate to estimate the option price (Hull, 2006).

The Monte Carlo simulation technique was introduced by Boyle (1977) to price European options. The Monte Carlo simulation technique can be used efficiently in situations where the payoff from the derivative depends on several underlying market variables. Additionally, it also estimates a standard error for the estimates made. Even if the Monte Carlo simulation procedure is very time consuming for a single variable, it is considered to be quicker compared to the other procedures for more variables since the total time taken by the Monte Carlo simulation increases linearly with the number of variables, while the total time taken by other procedures increases exponentially. Even though the Monte Carlo simulation procedure can not handle American options easily, Broadie *et al.* (1997) have proposed an enhanced Monte Carlo simulation technique for American options.

2.1.3 Finite difference methods

For valuing the derivatives, finite difference methods such as implicit and explicit finite difference methods (e.g., Hull and White, 1990; Wilmott, 1998) are available. Finite difference methods value a derivative by iteratively solving the differential equations that describe the behaviour of the underlying asset. Finite difference methods are applicable for both European and American options. However, the situation in which payoff depends on more than one underlying variable can not be dealt easily using finite difference methods.

2.1.4 Lattice approach

Since developing a lattice approach that takes into account skewness and kurtosis is the main purpose of this project, a more detailed explanation about the lattice approach has been presented in this section.

A lattice is a graphical representation of all the possible paths that might be followed by the stock price (Hull, 2006). In this technique, a life of an option is discretized into a large number of time intervals. In this discrete framework, the stock price movements are calculated at each small time interval from the present time to the expiration time. Then the option value is estimated by discounting backwards from the expiration to the present.

Cox *et al.* (1979) have developed a binomial lattice approach, shown in Figure 2.1. In a binomial lattice, let p be the probability that the value of variable goes up in time Δt and ϕ be the continuously compounded return of the underlying variable when it is going along the upward branch. These two parameters, ϕ and p, can be determined by matching the first two moments of the process implied by the binomial lattice with those of the continuous process over a small discrete time interval Δt.

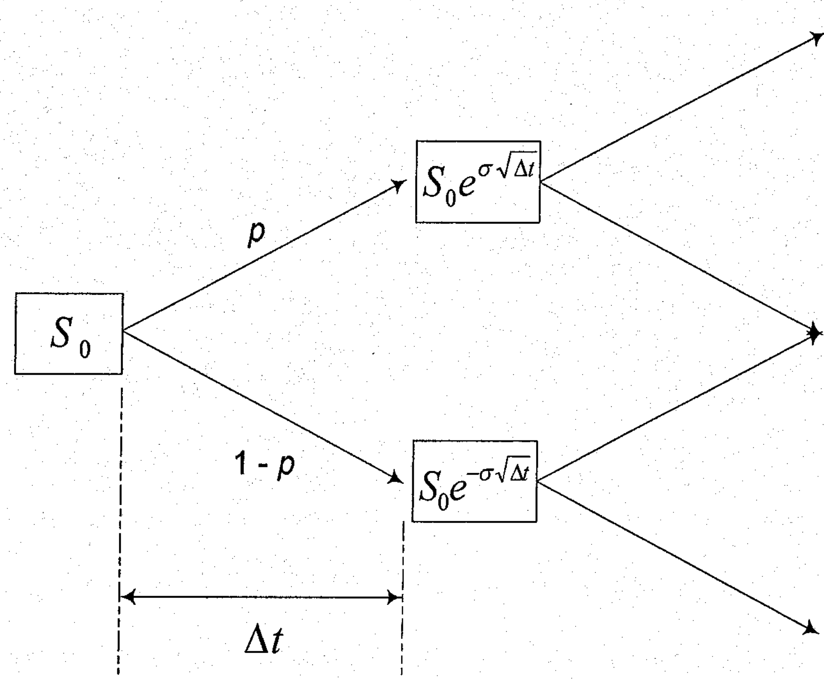


Figure 2.1 Binomial lattice model

Cox *et al.* (1979) have considered that a binomial lattice is governed by the geometric Brownian motion. Therefore, for a small time interval, Δt, the process implies that the continuously compounded return of that stochastic process is normally distributed with a mean of $\vartheta\Delta t$ and a volatility of $\sigma\sqrt{\Delta t}$. Hence, by matching the mean and variance of the return implied by the binomial lattice with those of the continuous stochastic variable, we get,

$$pe^{\phi} + (1-p)e^{-\phi} = e^{\vartheta\Delta t} \tag{2.10}$$

$$\text{and } p(\phi)^2 + (1-p)(-\phi)^2 = \sigma^2\Delta t. \tag{2.11}$$

By solving Equations (2.10) and (2.11), we get

$$p = \frac{(e^{\vartheta\Delta t} - e^{-\phi})}{(e^{\phi} - e^{-\phi})} \quad \text{and} \quad \phi = \sigma\sqrt{\Delta t}. \tag{2.12}$$

Boyle (1988) has developed a trinomial lattice for a single underlying stochastic process. Three branches (up, middle, and down) have been considered as shown in Figure 2.2.

Boyle (1988) obtained the probability distribution by matching the first two moments of the lattice with those of the lognormal distribution over a small interval of time. According to Boyle's trinomial lattice model, the continuously compounded returns of the underlying variable when it is going along the up branch is given by,

$$\phi = \lambda\sigma\sqrt{\Delta t}. \tag{2.13}$$

By numerical examples, Boyle (1988) has shown that λ, a scaler, should be greater than one for keeping all the probabilities positive. Boyle (1988) has extended his analysis for two underlying stochastic processes using five branches.

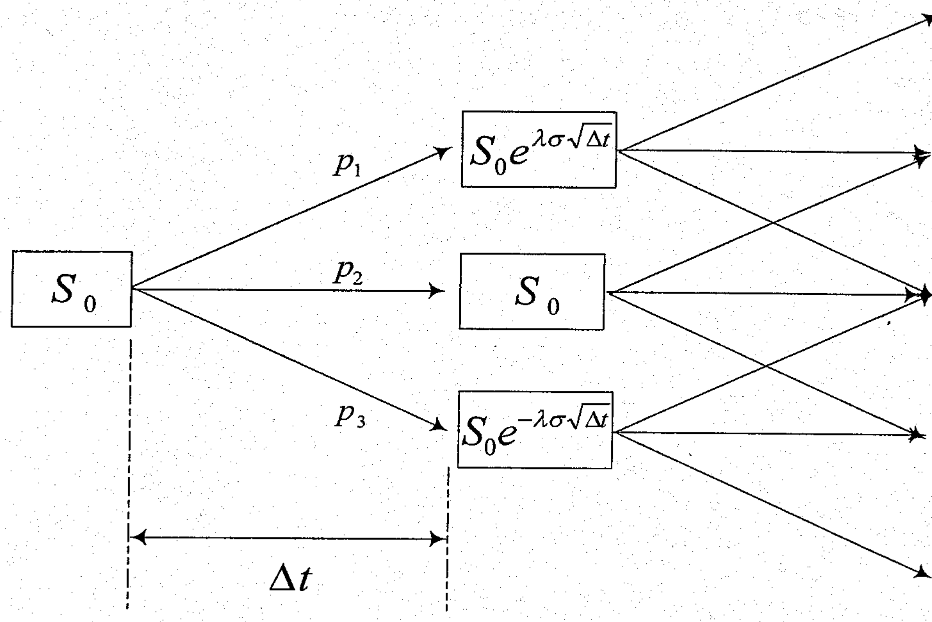


Figure 2.2 Trinomial lattice model

Boyle *et al.* (1989) have proposed a quadrinomial lattice for two underlying variables. The characteristic function of the discrete random variable has been matched with that of the continuous lognormal distribution. Boyle *et al.* (1989) have obtained positive probabilities for all branches having the condition

$$\phi = \sigma\sqrt{\Delta t}\,, \tag{2.14}$$

where ϕ represents the continuously compounded return of the underlying stochastic variable when it is going along the upward branch. The study has also been extended to n number of underlying variables.

Kamrad and Ritchken (1991) have proposed a trinomial lattice model for a single underlying variable. A pentanomial lattice model for two underlying variables has been proposed by matching the first two moments of the continuous normal distribution with those of discrete random variables over a short time interval. Kamrad and Ritchken (1991) have also proved that the trinomial lattice model converges faster than the binomial lattice model, and the pentanomial lattice model converges faster than the quadrinomial lattice.

2.1.5 Option pricing model that incorporates skewness and kurtosis

When the return on the stock is assumed to have a lognormal distribution, the third and fourth moments (skewness and kurtosis) must be considered in the lattice construction. However, the aforementioned lattice models have considered only the first two moments to develop the lattices. Rubinstein (1994) has proposed a lattice model that incorporates skewness and kurtosis. Primbs *et al.* (2007) have also proposed a pentanomial lattice model by matching the first four central moments of discrete random variable with those of a continuous random variable. By doing so, Primbs *et al.* (2007) have found the probability distribution with their positivity conditions in terms of skewness and kurtosis. As the time interval between two steps tends to zero, the discrete model converges to a continuous lattice framework. Hence, using the relationship between central moments and cumulants, for the small time interval, the probability distribution in terms of cumulants has been found; and then the positivity conditions have been obtained as follows:

$$c_4 c_2 \geq 3{c_3}^2 \text{ and } c_4 \geq 0, \tag{2.15}$$

where c_i is the i^{th} cumulant. Figure 2.3 shows the pentanomial lattice model developed in Primbs *et.al* (2007), where the distance between two outcomes (the amount by which the value goes up or down) was considered to be 2α, where α is a parameter given by,

$$\alpha = \frac{1}{2}\sqrt{c_2\tau + \frac{c_4}{3c_2}}\,. \tag{2.16}$$

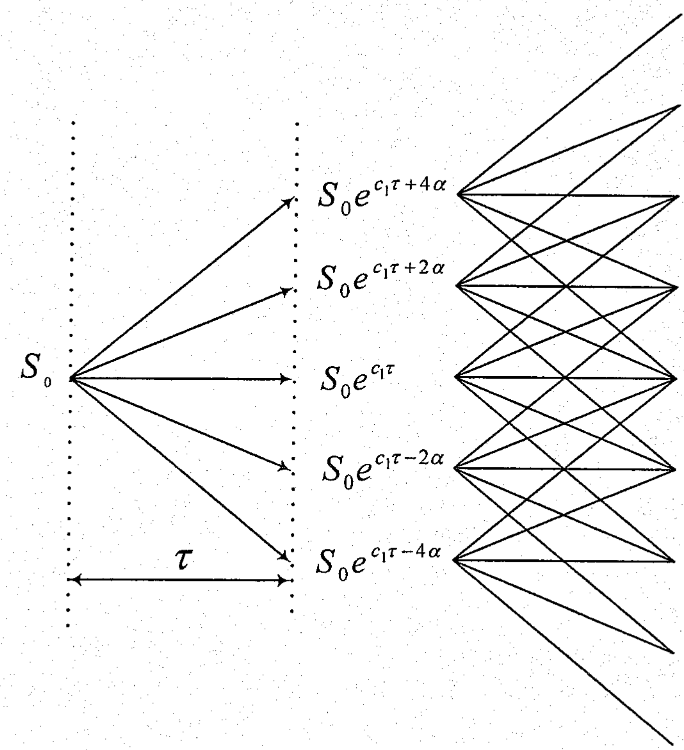


Figure 2.3 Pentanomial lattice model

Primbs *et al.* (2007) have claimed that the recombination conditions along with the requirement of non-negative probabilities are relatively restrictive in terms of skewness and kurtosis for the quadrinomial lattice, which is not true. In the next chapter, we have developed a quadrinomial lattice that includes skewness and kurtosis and derived the same conditions given in Equation (2.15), which have been obtained using the pentanomial lattice model.

CHAPTER 3

This chapter presents a procedure for developing a quadrinomial lattice model incorporating skewness and kurtosis along with the first two moments. The procedure is to create a discrete random variable whose first four central moments match those of a continuous variable. Then a lattice has been constructed using the relationship between central moments and cumulants. Finally, for a small time interval, the recombination conditions along with positive branch probabilities have been obtained in terms of the skewness and kurtosis.

3.1 Problem Definition

Primbs *et al.* (2007) have developed a pentanomial lattice model that includes skewness and kurtosis and also obtained the recombination conditions along with positive probabilities in terms of skewness and kurtosis. However, it has been claimed that *"With four moments, one could conceivably use a quadrinomial lattice (i.e., four branches); however, the recombination conditions along with the requirement of non-negative probabilities are quite limiting in terms of the range of skewness and kurtosis that can be captured."* In this chapter, we have developed a quadrinomial lattice model that includes both skewness and kurtosis and obtained the conditions for non-negative probabilities that are the same as the one obtained in Primbs *et al.* (2007) for the pentanomial lattice model.

3.2 Development of a Quadrinomial Lattice Model

Let the stock price follow the exponential Levy process that is defined as

$$S_t = S_0 e^{X_t}. \quad (3.1)$$

First, we will construct the lattice for X_t, and then by simply taking the exponential of X_t, we will develop the lattice model for S_t. In order to create the lattice for X (for simplicity, the time index t is omitted), we will define a discrete random variable Z; and then by matching the first four moments of the discrete random variable, Z, with those of continuous variable, X, we can construct the lattice for X.

3.2.1 Parameterization of quadrinomial lattice model

First, consider the random variable X. Let us say m_j is its j^{th} raw moment, μ_j is its j^{th} central moment and c_j is its j^{th} cumulant. Now define Z, the discrete random variable, having values

$$Z = m_1 + (L - 1 - l)\alpha, \quad l = 1,2,3,4 \text{ with probabilities } p_l\,, \quad (3.2)$$

where α is a parameter that shows the jump size (distance between two outcomes), m_1 is the mean of X, L is equal to number of branches, and in this case $L = 4$. Figure 3.1 shows the possible values of the discrete random variable Z.

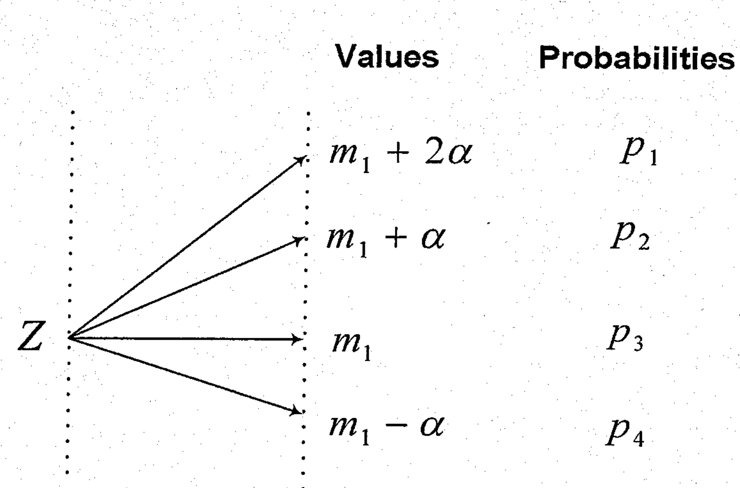


Figure 3.1 Possible values of Z with their probabilities

Now we need to match the first four moments of Z with those of X to consider skewness and kurtosis. Therefore,

$$(2\alpha)\, p_1 + (\alpha)\, p_2 + (0)\, p_3 + (-\alpha)\, p_4 = \mu_1 \,, \tag{3.3}$$

$$(2\alpha)^2\, p_1 + (\alpha)^2\, p_2 + (0)^2\, p_3 + (-\alpha)^2\, p_4 = \mu_2 \,, \tag{3.4}$$

$$(2\alpha)^3\, p_1 + (\alpha)^3\, p_2 + (0)^3\, p_3 + (-\alpha)^3\, p_4 = \mu_3 \,, \tag{3.5}$$

$$(2\alpha)^4\, p_1 + (\alpha)^4\, p_2 + (0)^4\, p_3 + (-\alpha)^4\, p_4 = \mu_4 \,, \tag{3.6}$$

and since all probabilities sum up to one, we have

$$p_1 + p_2 + p_3 + p_4 = 1 \,. \tag{3.7}$$

Now solving Equations (3.4), (3.5), (3.6), and (3.7), we get

$$\begin{aligned} p_1 &= \frac{\mu_4 - \mu_2\alpha^2}{12\alpha^4} \,, \\ p_2 &= \frac{-\mu_4 + \mu_3\alpha + 2\mu_2\alpha^2}{2\alpha^4} \,, \\ p_3 &= 1 + \frac{\mu_4 - 5\mu_2\alpha^2}{4\alpha^4} \,, \\ p_4 &= \frac{\mu_4 - 3\mu_3\alpha + 2\mu_2\alpha^2}{6\alpha^4} \,. \end{aligned} \tag{3.8}$$

The detailed derivation of Equation (3.8) has been shown in Appendix A.

These probabilities are in terms of the second, third, and fourth central moments. By substituting $p_1, p_2, p_3,$ and p_4, given in Equation (3.8), in Equation (3.3) and solving it for α, it will give us the jump size in terms of central moments.

Therefore, by substituting Equation (3.8) in Equation (3.3), we get,

$$(2\alpha)\left[\frac{\mu_4-\mu_2\alpha^2}{12\alpha^4}\right]+(\alpha)\left[\frac{-\mu_4+\mu_3\alpha+2\mu_2\alpha^2}{2\alpha^4}\right]+(-\alpha)\left[\frac{\mu_4-3\mu_3\alpha+2\mu_2\alpha^2}{6\alpha^4}\right]=\mu_1. \tag{3.9}$$

By simplifying Equation (3.9), we get,

$$\mu_4-2\mu_3\alpha-\mu_2\alpha^2+2\mu_1\alpha^3=0\,. \tag{3.10}$$

Since we are matching the moments of the discrete random variable Z with those of the random variable X, and m_1 is the mean of X, the condition $\mu_1=0$ must be satisfied to make sure that the mean of Z remains m_1.

So by replacing $\mu_1=0$ in Equation (3.10), we get,

$$\mu_4-2\mu_3\alpha-\mu_2\alpha^2=0. \tag{3.11}$$

Now by solving Equation (3.11) for α, it leads us to

$$\begin{aligned}\alpha&=\frac{-\mu_3-\sqrt{\mu_3{}^2+\mu_2\mu_4}}{\mu_2}\\ \text{or}\quad \alpha&=\frac{-\mu_3+\sqrt{\mu_3{}^2+\mu_2\mu_4}}{\mu_2}.\end{aligned} \tag{3.12}$$

Since the fourth central moment is the expectation of a fourth power, it is always strictly positive, i.e., $\mu_4\geq 0$. Hence, $\sqrt{\mu_3{}^2+\mu_2\mu_4}\geq 0$ and $\sqrt{\mu_3{}^2+\mu_2\mu_4}\geq\mu_3$.

Therefore, $$\alpha=\frac{-\mu_3+\sqrt{\mu_3{}^2+\mu_2\mu_4}}{\mu_2}\geq 0.$$

Now consider

$$\alpha = \frac{-\mu_3 - \sqrt{{\mu_3}^2 + \mu_2\mu_4}}{\mu_2} < 0,$$

i.e., α will be negative. In this case, simply inverting the lattice (two branches downward, one central branch and one branch upward) will result in positive α.

For simplicity, consider

$$\alpha = \frac{-\mu_3 + \sqrt{{\mu_3}^2 + \mu_2\mu_4}}{\mu_2}. \tag{3.13}$$

The general procedure for moment matching has been described so far. Now in the next section, the lattice construction using a discretized framework will be shown.

3.1.2 Creating lattice model

Now for any given time period t, we will use the results obtained so far and we will match the moments of X_t with those of $Z(t)$. As indicated in Equation (3.1), X_t is a Levy process, so its cumulants are linearly proportional to time (Primbs *et al.*, 2007). Since the yearly cumulants are specified, cumulants at any time t can be estimated easily. If, let us say c_j is the j^{th} cumulant of X_1, then the j^{th} cumulant of X_t will be $c_j t$. The concept of cumulants was introduced for more than two moments as this method is simpler compared to the method in which the central moments for a lognormal distribution are estimated and then moments are matched with those of a discrete variable. The derivation of the first four central moments of the lognormal distribution is shown in Appendix B.

Before we go any further, it is appropriate to give the relationship between central moments and cumulants.

$$\mu_2 = c_2, \tag{3.14}$$

$$\mu_3 = c_3, \tag{3.15}$$

$$\mu_4 = c_4 + 3{c_2}^2. \tag{3.16}$$

Now if we have N step increments in t, then the time period, τ, between two steps is

$$\tau = \frac{t}{N}. \tag{3.17}$$

Hence, to create the lattice model for X_t, we first need to match each increment X_τ with the discrete random variable $Z(\tau)$, which corresponds to the recombining lattice model.

Let us say S_0 is the initial stock price, then the model $S_t = S_0 e^{X_t}$ is given by

$$S_t = S_0 e^{\sum_{k=1}^{N}(Z_k(\tau))}, \tag{3.18}$$

where $Z_k(\tau)$ is the random variable expressed in terms of cumulants as

$$Z_k(\tau) = \begin{cases} c_1\tau + 2\alpha & p_1(\tau) = \dfrac{c_4\tau + 3{c_2}^2\tau^2 - c_2\tau\alpha^2}{12\alpha^4}, \\ c_1\tau + \alpha & p_2(\tau) = \dfrac{-c_4\tau - 3{c_2}^2\tau^2 + c_3\tau\alpha + 2c_2\tau\alpha^2}{2\alpha^4}, \\ c_1\tau & p_3(\tau) = 1 + \dfrac{c_4\tau + 3{c_2}^2\tau^2 - 5c_2\tau\alpha^2}{4\alpha^4}, \\ c_1\tau - \alpha & p_4(\tau) = \dfrac{c_4\tau + 3{c_2}^2\tau^2 - 3c_3\tau\alpha + 2c_2\tau\alpha^2}{6\alpha^4}, \end{cases} \tag{3.19}$$

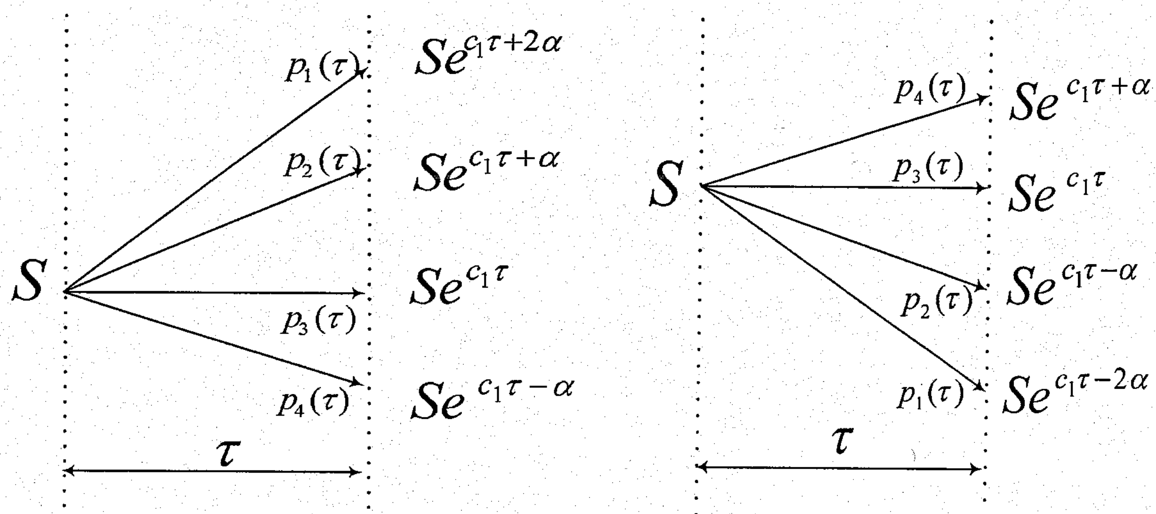
and also the jump size, α, in terms of cumulants can be written as

$$\alpha = \frac{-c_3\tau + \sqrt{{c_3}^2\tau^2 + c_2\tau(c_4\tau + 3{c_2}^2\tau^2)}}{c_2\tau}. \tag{3.20}$$

Therefore,

$$\alpha = \frac{-c_3 + \sqrt{{c_3}^2 + c_2(c_4 + 3{c_2}^2\tau)}}{c_2}. \tag{3.21}$$

Equations (3.19) and (3.20) are obtained from (3.8) and (3.13), respectively. Figure 3.2 shows the quadrinomial lattice model. Figure 3.2 (a) shows the model when $\alpha > 0$. When $\alpha < 0$, simply by inverting the branch as shown in Figure 3.2 (b), one could easily use the quadrinomial lattice model.



(a) The quadrinomial lattice with two up, one middle and one down branch

(b) The quadrinomial lattice with one up, one middle and two down branches

Figure 3.2 One step of the quadrinomial lattice

3.1.3 Limits of quadrinomial lattice model

Now as $\tau \to 0$, the lattice can be categorized as a continuous lattice model and, for that situation, we will obtain the conditions for which the probabilities, specified in Equation (3.19), remain positive.

So as $\tau \to 0$, from Equation (3.21), we get,

$$\alpha_0 = \frac{-c_3 + \sqrt{c_3{}^2 + c_2 c_4}}{c_2}. \tag{3.22}$$

Let us define ξ and the new probabilities q_1, q_2, and q_4 in terms of the branch probabilities defined in Equation (3.19) as follows:

$$\begin{aligned}
&\lim_{\tau\to 0}\left(\frac{1}{\tau}\right)p_1(\tau) = \frac{c_4 - c_2{\alpha_0}^2}{12{\alpha_0}^4} = \xi q_1,\\
&\lim_{\tau\to 0}\left(\frac{1}{\tau}\right)p_2(\tau) = \frac{-c_4 + c_3\alpha_0 + 2c_2{\alpha_0}^2}{2{\alpha_0}^4} = \xi q_2,\\
&\lim_{\tau\to 0}\left(\frac{1}{\tau}\right)(p_3(\tau)-1) = \frac{c_4 - 5c_2{\alpha_0}^2}{4{\alpha_0}^4} = -\xi,\\
&\lim_{\tau\to 0}\left(\frac{1}{\tau}\right)p_4(\tau) = \frac{c_4 - 3c_3\alpha_0 + 2c_2{\alpha_0}^2}{6{\alpha_0}^4} = \xi q_4.
\end{aligned} \tag{3.23}$$

Therefore,

$$q_1 + q_2 + q_4 = 1\,. \tag{3.24}$$

By substituting α_0 from Equation (3.22) in Equation (3.23) and solving for ξ, q_1, q_2, and q_4, we get,

$$\begin{aligned}
\xi &= \frac{{c_2}^3\left(5{c_3}^2 + 2c_2c_4 - 5c_3\sqrt{{c_3}^2 + c_2c_4}\right)}{2\left(c_3 - \sqrt{{c_3}^2 + c_2c_4}\right)^4},\\
q_1 &= \frac{c_3\left(c_3 - \sqrt{{c_3}^2 + c_2c_4}\right)}{3\left(-5{c_3}^2 - 2c_2c_4 + 5c_3\sqrt{{c_3}^2 + c_2c_4}\right)},\\
q_2 &= \frac{3{c_3}^2 + c_2c_4 - 3c_3\sqrt{{c_3}^2 + c_2c_4}}{5{c_3}^2 + 2c_2c_4 - 5c_3\sqrt{{c_3}^2 + c_2c_4}},\\
q_4 &= \frac{7{c_3}^2 + 3c_2c_4 - 7c_3\sqrt{{c_3}^2 + c_2c_4}}{15{c_3}^2 + 6c_2c_4 - 15c_3\sqrt{{c_3}^2 + c_2c_4}}.
\end{aligned} \tag{3.25}$$

From Equation (3.25), q_1, q_2, and q_4 can also be written in terms of ξ, that is,

$$\begin{aligned}
\xi &= \frac{c_2{}^3}{2}\left[\frac{2c_2c_4}{\left(\sqrt{c_3{}^2+c_2c_4}-c_3\right)^4}-\frac{5c_3}{\left(\sqrt{c_3{}^2+c_2c_4}-c_3\right)^3}\right],\\
q_1 &= \frac{c_2{}^3}{6\xi}\left[\frac{c_3}{\left(\sqrt{c_3{}^2+c_2c_4}-c_3\right)^3}\right],\\
q_2 &= \frac{c_2{}^3}{6\xi}\left[\frac{3c_2c_4}{\left(\sqrt{c_3{}^2+c_2c_4}-c_3\right)^4}-\frac{9c_3}{\left(\sqrt{c_3{}^2+c_2c_4}-c_3\right)^3}\right],\\
q_4 &= \frac{c_2{}^3}{6\xi}\left[\frac{3c_2c_4}{\left(\sqrt{c_3{}^2+c_2c_4}-c_3\right)^4}-\frac{7c_3}{\left(\sqrt{c_3{}^2+c_2c_4}-c_3\right)^3}\right].
\end{aligned} \tag{3.26}$$

To make sure the probabilities specified in Equation (3.26) remain positive, the following conditions should be satisfied.

1) For ξ to be positive:

$$\frac{2c_2c_4}{\left(\sqrt{c_3{}^2+c_2c_4}-c_3\right)^4} \geq \frac{5c_3}{\left(\sqrt{c_3{}^2+c_2c_4}-c_3\right)^3}. \tag{3.27}$$

Simplifying the above equation, we get,

$$\frac{2c_2c_4+5c_3{}^2}{5c_3} \geq \sqrt{c_3{}^2+c_2c_4}\,. \tag{3.28}$$

Squaring both sides,

$$4c_2{}^2c_4{}^2+25c_3{}^4+20c_2c_4c_3{}^2 \geq 25c_3{}^4+25c_2c_4c_3{}^2. \tag{3.29}$$

Therefore,

$$c_2c_4 \geq \left(\frac{5}{4}\right)c_3{}^2. \tag{3.30}$$

2) For q_1 to be positive:

$$\sqrt{c_3{}^2 + c_2c_4} \geq c_3 \quad \text{and} \quad c_3 \geq 0\,. \tag{3.31}$$

Simplifying Equation (3.31), we get

$$c_4 \geq 0 \quad \text{and} \quad c_3 \geq 0\,. \tag{3.32}$$

3) For q_2 to be positive:

$$\frac{3c_2c_4}{\left(\sqrt{c_3{}^2 + c_2c_4} - c_3\right)^4} \geq \frac{9c_3}{\left(\sqrt{c_3{}^2 + c_2c_4} - c_3\right)^3}, \tag{3.33}$$

which simplifies to,

$$c_2c_4 \geq 3c_3{}^2. \tag{3.34}$$

4) For q_4 to remain positive:

$$\frac{3c_2c_4}{\left(\sqrt{c_3{}^2 + c_2c_4} - c_3\right)^4} \geq \frac{7c_3}{\left(\sqrt{c_3{}^2 + c_2c_4} - c_3\right)^3}, \tag{3.35}$$

which leads to,

$$c_2c_4 \geq \left(\frac{7}{9}\right)c_3{}^2. \tag{3.36}$$

Among all the conditions given in Equations (3.30), (3.34), and (3.36), the most constraining condition is,

$$c_2c_4 \geq 3c_3{}^2$$

Therefore, there are three conditions that must be satisfied for non-negative probability distributions:

$$c_2 c_4 \geq 3{c_3}^2, \qquad c_4 \geq 0, \quad \text{and} \quad c_3 \geq 0. \tag{3.37}$$

Now that we have derived these probabilities and conditions, consider Figure 3.2 (a) having the jump size, α, given in Equation (3.12), equal to

$$\alpha = \frac{-\mu_3 + \sqrt{{\mu_3}^2 + \mu_2\mu_4}}{\mu_2}.$$

This corresponds to a positively skewed lattice and that is the reason we have $c_3 \geq 0$ in Equation (3.37).

For a negatively skewed lattice, by selecting

$$\alpha = \frac{-\mu_3 - \sqrt{{\mu_3}^2 + \mu_2\mu_4}}{\mu_2},$$

the conditions that can be obtained are:

$$c_2 c_4 \geq 3{c_3}^2, \qquad c_4 \geq 0, \quad \text{and} \quad c_3 \leq 0. \tag{3.38}$$

Appendix C contains the derivation for the negatively skewed lattice (Figure 3.2 (b)) and shows the conditions given in Equation (3.38).

Hence, for the given skewness (positive or negative), by appropriately choosing α and properly constructing one of the lattices shown in Figure 3.2, one can use the developed quadrinomial lattice model. Hence, from Equations (3.37) and (3.38), the conditions $c_3 \geq 0$ and $c_3 \leq 0$ can be omitted respectively.

Therefore, the two conditions, for which the developed quadrinomial lattice is applicable, are:

$$c_4 c_2 \geq 3{c_3}^2 \text{ and } c_4 \geq 0 . \tag{3.39}$$

The conditions specified in Equation (3.39) for non-negative probabilities are the same as the one obtained in Primbs *et al.* (2007) for the pentanomial lattice model (Equation (2.15)).

The next chapter contains numerical examples and compares the convergence rate, the effect of volatility and the volatility smiles and smirks of the quadrinomial lattice model developed in this chapter with those of the pentanomial lattice model developed in Primbs *et al.* (2007).

CHAPTER 4

In this chapter, first, we will demonstrate how to use the quadrinomial lattice, developed in Chapter 3, for option pricing. Then, we will show the convergence of the quadrinomial lattice in comparison with that of the pentanomial lattice proposed in Primbs *et al.* (2007). Finally, we will compare the effect of the volatility, the volatility smiles and smirks generated by the pentanomial lattice and the quadrinomial lattice.

4.1 Option Pricing using the Quadrinomial Lattice

In this section, we will show the features of the quadrinomial lattice model for option pricing considering a case that involves a non-dividend paying underlying asset with daily skewness and excess kurtosis of

$$\varsigma = 0.5, \quad k = 3$$

The yearly volatility is $\sigma = 0.2$; the risk-free rate, r_f, and the first cumulant, c_1, are assumed to be zero. Total trading days are assumed to be 250 and we have used time to expiration, T, of 20 days. The aforementioned values of parameters are same as the one used in Primbs *et al.* (2007). Since the initial stock price and strike price were not mentioned, we assume that the initial stock price, S_0, to be \$100 and the strike price, ϒ, is to be \$100. That is, we use this model to price the at-the-money European call option.

Since the daily skewness and excess kurtosis are given, we need to convert them into yearly skewness and excess kurtosis respectively.

If the cumulants are considered as independent parameters, they possess the additive property when summed together. Therefore,

$$j^{th} \text{ order cumulants of the } N \text{ day log stock return} = NC_j, \quad j = 1, 2, \ldots \tag{4.1}$$

Now daily skewness and excess kurtosis can be expressed, in terms of cumulants, as

$$\varsigma = \frac{c_3}{c_2^{3/2}}, \qquad k = \frac{c_4}{c_2^{2}}. \tag{4.2}$$

Combining Equations (4.1) and (4.2), it leads to,

$$N\ day\ skewness = \frac{Nc_3}{(Nc_2)^{3/2}} = \frac{\varsigma}{N^{1/2}}, \tag{4.3}$$

$$N\ day\ kurtosis = \frac{Nc_4}{(Nc_2)^2} = \frac{k}{N}. \tag{4.4}$$

So if we denote yearly skewness and excess kurtosis as $\hat{\varsigma}$ and $\hat{k}$ respectively, by considering a total of 250 trading days in a year, we get,

$$\hat{\varsigma} = \frac{0.5}{(250)^{1/2}} = 0.0316,$$

$$\hat{k} = \frac{3}{250} = 0.012\,.$$

From volatility, the second cumulant can be calculated as follows:

$$c_2 = \sigma^2 = 0.2^2 = 0.04\,.$$

From the definition of skewness,

$$c_3 = \hat{\varsigma}\, c_2^{3/2} = (0.0316)(0.04)^{3/2} = 2.5298 \times 10^{-4}\,.$$

From the definition of kurtosis, we can have the fourth cumulant,

$$c_4 = \hat{k}\, c_2^{\,2} = (0.012)(0.04)^2 = 1.92 \times 10^{-5}\,.$$

We consider a two-step lattice to price a European call option with a maturity period of 20 days. Hence, the time step-size is 10 days. It can be expressed in terms of years as follows:

$$\text{Time period between two steps } (\tau) = \frac{T}{\text{No of steps}} = \frac{\left(20/250\right)}{2} = 0.04 \text{ years}\,.$$

Figure 4.1 shows a two-step quadrinomial lattice model developed in Chapter 3 with $c_3 \geq 0$. As shown in Figure 4.1, at time zero, the initial stock price is S_0. Since we are using the quadrinomial lattice, after time period τ, we can have four possible nodes that can be obtained by extending two upward, one middle and one downward branches from the initial node. Similarly, after time period 2τ, seven nodes can be generated in the same pattern.

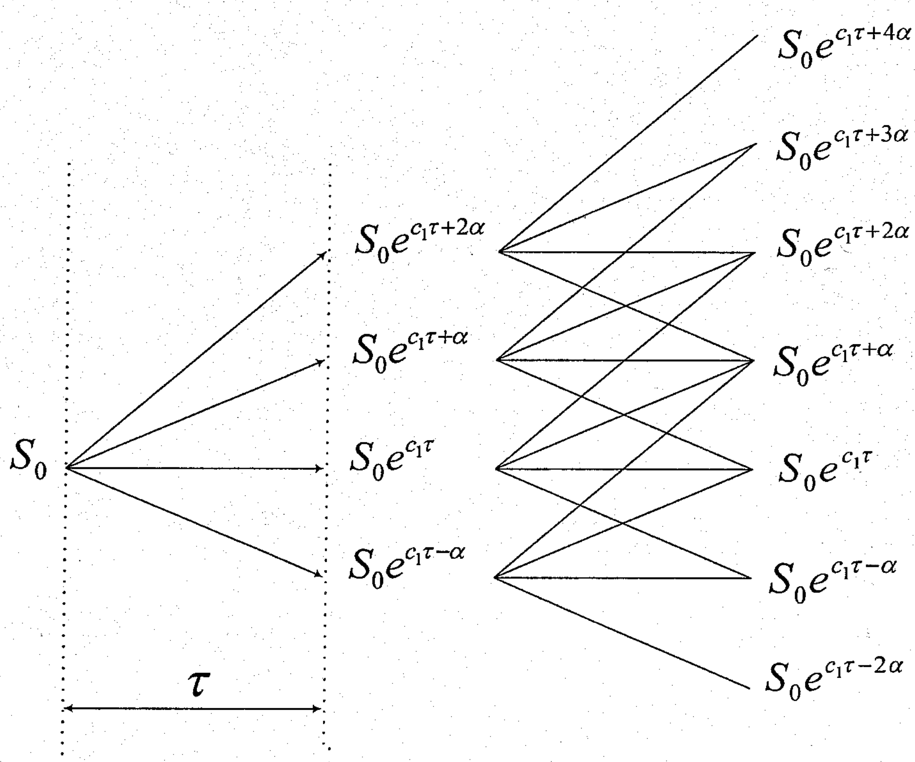


Figure 4.1 A two-step quadrinomial lattice model with $c_3 \geq 0$.

Now, for each time period τ, the jump size, α, can be calculated using Equation (3.21),

$$\alpha = \frac{-c_3 + \sqrt{c_3{}^2 + c_2(c_4 + 3c_2{}^2\tau)}}{c_2}$$

$$\alpha = \frac{-(2.5298 \times 10^{-4}) + \sqrt{(2.5298 \times 10^{-4})^2 + (0.04)\{1.92 \times 10^{-5} + 3(0.04)^2(0.04)\}}}{0.04}.$$

Therefore, $\alpha = 0.0666.$

Then probability for each branch can be calculated using Equation (3.8),

$$p_1 = \frac{c_4\tau + 3c_2{}^2\tau^2 - c_2\tau\alpha^2}{12\alpha^4}$$

$$= \frac{(1.92 \times 10^{-5} \times 0.04) + [3 \times (0.04)^2 \times (0.04)^2] - [0.04 \times 0.04 \times (0.0666)^2]}{12 \times (0.0666)^4}$$

$$= 0.0057,$$

$$p_2 = \frac{-c_4\tau - 3c_2{}^2\tau^2 + c_3\tau\alpha + 2c_2\tau\alpha^2}{2\alpha^4}$$

$$= \{[-(1.92 \times 10^{-5}) \times 0.04] - [3 \times (0.04)^2 \times (0.04)^2] + [(2.5298 \times 10^{-4}) \times 0.04 \times 0.0666] + [2 \times 0.04 \times 0.04 \times (0.0666)^2]\}/[2 \times (0.0666)^4]$$

$$= 0.1632,$$

$$p_3 = 1 + \frac{c_4\tau + 3c_2{}^2\tau^2 - 5c_2\tau\alpha^2}{4\alpha^4}$$

$$= 1 + \frac{(1.92 \times 10^{-5} \times 0.04) + [3 \times (0.04)^2 \times (0.04)^2] - [5 \times 0.04 \times 0.04 \times (0.0666)^2]}{4 \times (0.0666)^4}$$

$$= 0.6565,$$

$$p_4 = \frac{c_4\tau + 3{c_2}^2\tau^2 - 3c_3\tau\alpha + 2c_2\tau\alpha^2}{6\alpha^4}$$

$$= \{(1.92 \times 10^{-5} \times 0.04) + [3 \times (0.04)^2 \times (0.04)^2] - (3 \times 2.5298 \times 10^{-4} \times 0.04 \times 0.0666) + [2 \times 0.04 \times 0.04 \times (0.0666)^2]\}/[6 \times (0.0666)^4]$$

$$= 0.1746.$$

Figure 4.2 shows a two-step quadrinomial lattice with both the stock price (the upper number at each node) and option price (the lower number at each node). Once the price step-size, α, and time step-size, τ, are known, the stock price at each node can be determined. For example, let us consider node C, F, and K.

At node C:

Stock price at node C = (Stock price at node A) $\times\, e^{c_1\tau+\alpha} = 100 \times e^{0.0666} = \$\,106.8883$.

At node F:

Stock price at node F = (Stock price at node B) $\times\, e^{c_1\tau+2\alpha} = 114.2510 \times e^{2\times 0.0666}$

$= \$\,130.5329$.

At node K:

Stock price at node K = (Stock price at node E) $\times\, e^{c_1\tau} = 93.5557 \times e^{0.0000} = \$\,93.5557$

Or

Stock price at node K = (Stock price at node D) $\times\, e^{c_1\tau-\alpha} = 100 \times e^{-0.0666} = \$\,93.5557$.

Since we know the stock price at each node, first the payoffs can be determined at the last layer of the lattice and then the discounted expected value at the seed node can be found to determine the option price. As it is an at-the-money option, the strike price Y = $100. For the European call option, at nodes F, G, H, I, J, K, and L, we can calculate the option price simply by using the following formula:

$$\text{Call option price at given node} = \max(\text{stock price at given node} - \text{strike price},\ \ 0).$$

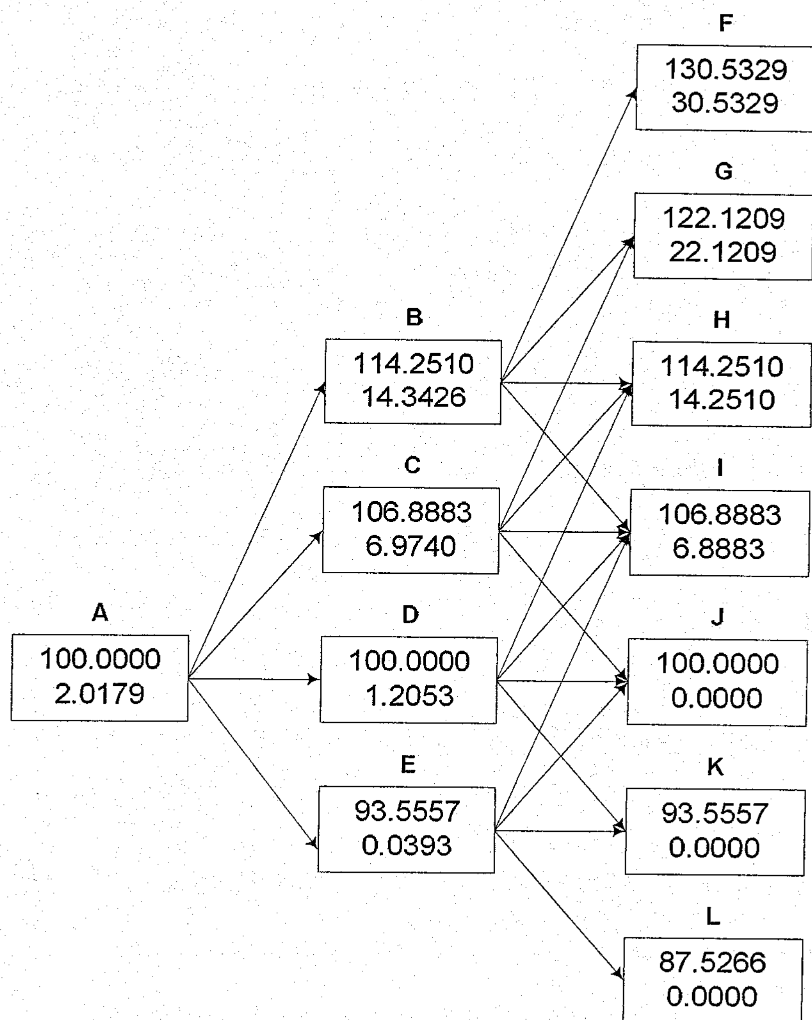


Figure 4.2 Two steps of the quadrinomial lattice for maturity period of 20 days

Sample calculations:

At node H:

$$\text{Call option price at H} = \max(\text{stock price at H} - \text{strike price},\ \ 0).$$

Therefore, Call option price at H $= \max\ (114.2510 - 100,\ 0) = \$\ 14.2510.$

Now discounting backward, we can calculate the option prices for the remaining nodes as follows:

Consider *node C:*

$$\text{Call option price at C} = \begin{Bmatrix} (p_1 \times \text{option price at G}) + (p_2 \times \text{option price at H}) \\ +(p_3 \times \text{option price at I}) + (p_4 \times \text{option price at J}) \end{Bmatrix} \times e^{-\tau r_f}$$

$$\text{Call option price at C} = \begin{Bmatrix} (0.0057 \times 22.1209) + (0.1632 \times 14.2510) \\ +(0.6565 \times 6.8883) + (0.1746 \times 0.0000) \end{Bmatrix} \times e^{-0.04 \times 0.0}$$

Therefore, Call option price at C $= \$\ 6.9740.$

The option values can be calculated the same way for nodes B, D, and E. The present value of the option is obtained by discounting them back at node A as follows:

At node A:

$$\text{Call option price at A} = \begin{Bmatrix} (p_1 \times \text{option price at B}) + (p_2 \times \text{option price at C}) \\ +(p_3 \times \text{option price at D}) + (p_4 \times \text{option price at E}) \end{Bmatrix} \times e^{-\tau r_f}$$

$$\text{Call option price at A} = \begin{Bmatrix} (0.0057 \times 14.3426) + (0.1632 \times 6.9740) \\ +(0.6565 \times 1.2053) + (0.1746 \times 0.0393) \end{Bmatrix} \times e^{-0.04 \times 0.0}$$

Therefore, Call option price at A $= \$\ \mathbf{2.0179}.$

So the option value of the at-the-money European call option that has 20 days of maturity is $2.0179 using the quadrinomial lattice model.

4.2 Convergence of Quadrinomial Lattice

In this section, we have compared the convergence rate of the quadrinomial lattice with that of the pentanomial lattice.

The at-the-money European call option of a non-dividend paying underlying asset has been valued. In this example, we have considered a maturity period of 1 year (250 days) and the rest of the parameters are the same as above. The number of steps is varied from 10 to 250 steps in intervals of 10. For example, 250 steps with a maturity period of 250 days means 1 day is considered as one time step size. Two different cases have been considered for two different values of the daily skewness: (1) $\varsigma = 0.5$ (Figure 4.3) and (2) $\varsigma = 0$ (Figure 4.4).

Case 1: Daily skewness $\varsigma = 0.5$

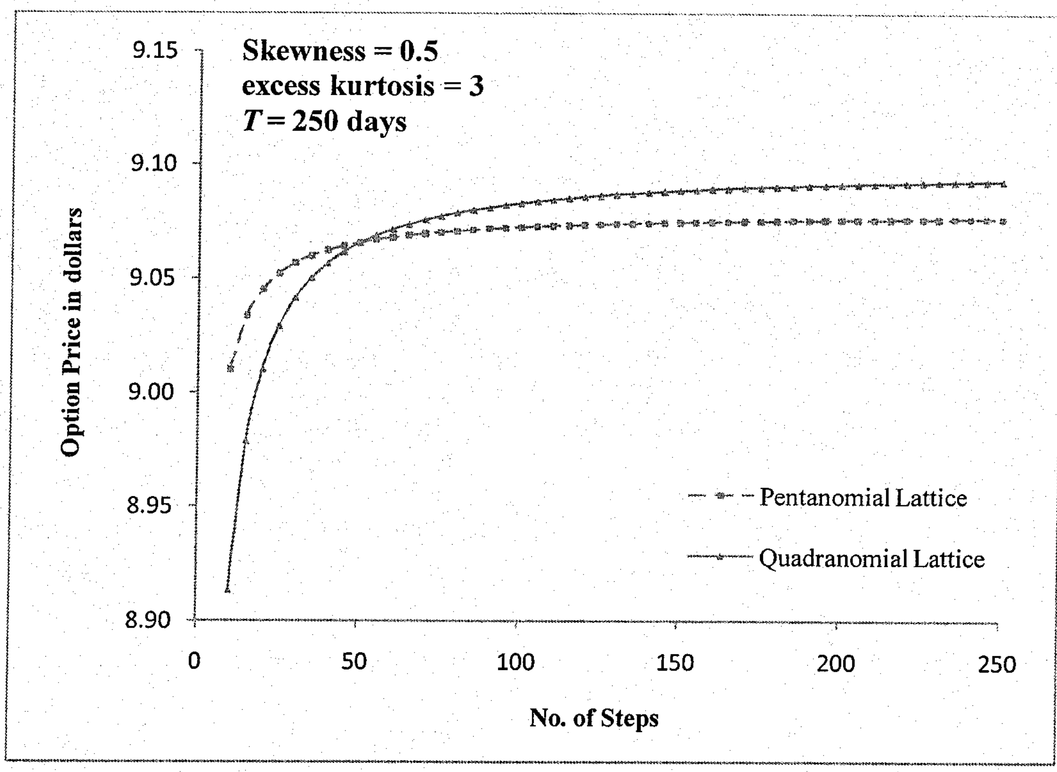


Figure 4.3 Convergence rates of pentanomial and quadrinomial models for an at-the-money call option with daily skewness equal to 0.5.

Case 2: Daily skewness $\varsigma = 0$

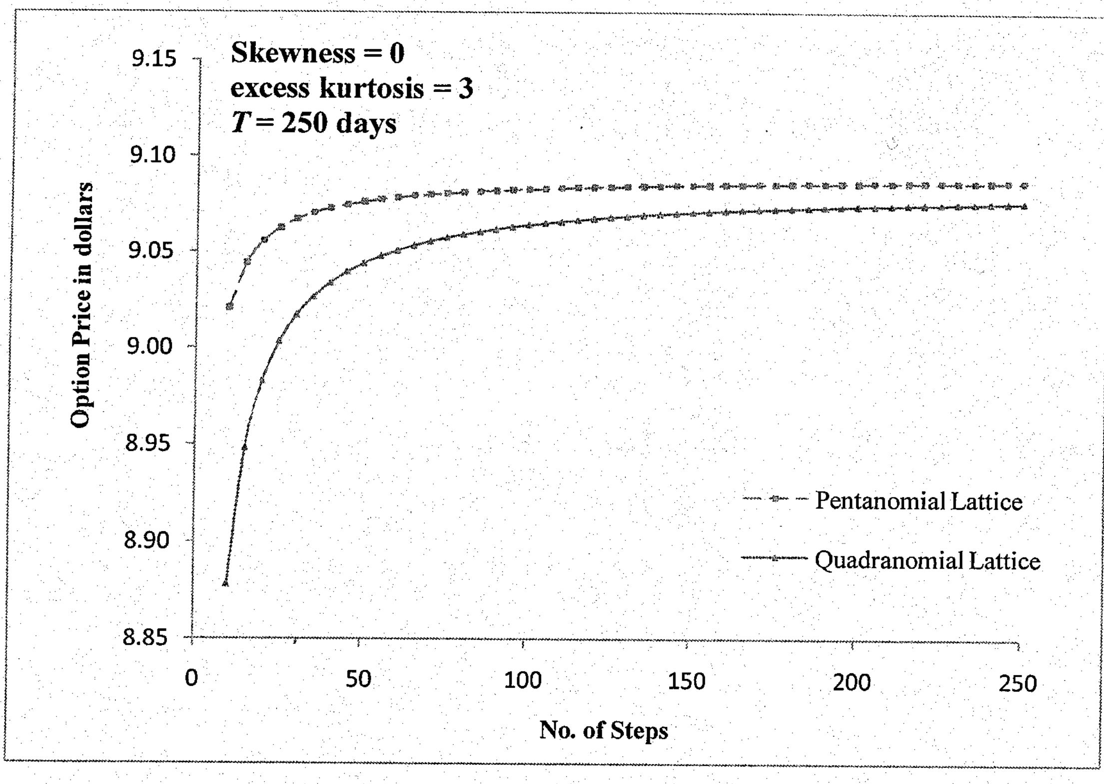


Figure 4.4 Convergence rates of pentanomial and quadrinomial models for an at-the-money call option with daily skewness equal to 0.

In case 1, as shown in Figure 4.3, after 150 steps the quadrinomial lattice converges while after 100 steps the pentanomial lattice converges. As expected, the higher number of branches in a lattice, the faster the convergence rate. One could observe that, after 50 steps, the price determined by the quadrinomial lattice is a bit higher than the price determined by the pentanomial lattice. However, the difference in the option prices obtained by the pentanomial lattice and the quadrinomial lattice is less than 2 cents for a maturity period of 1 year with 250 steps.

In case 2, as shown in Figure 4.4, after 200 steps the quadrinomial lattice converges, while after 100 steps the pentanomial lattice converges. The reason is that the higher number of branches in a lattice, the faster the convergence rate. The price determined by the quadrinomial lattice is lower than that of the pentanomial lattice. However, the difference between the prices is getting smaller as the steps increases. In this case also, the difference between the option prices obtained by the pentanomial and the quadrinomial lattices is less than 2 cents for maturity period of 1 year with 250 steps.

Therefore, from Figure 4.3 and 4.4, it can be said that the quadrinomial lattice model can also be used efficiently and effectively since the difference in the option prices, obtained from the quadrinomial lattice and the pentanomial lattice, is not much.

4.3 Effect of Volatility on Quadrinomial Lattice

This section shows the effect of higher volatility on the quadrinomial lattice model (developed in Chapter 3) compared to the pentanomial lattice model (proposed in Primbs *et al.* (2007)).

Again, the at-the-money European call option of a non-dividend paying underlying asset has been considered with the same parameters used in section 4.1 and 4.2. Both the quadrinomial and the pentanomial lattices are used to determine the option price assuming maturity period of 250 days (1 year) with 100 time steps. Case 1 is for the daily skewness of 0.5 (Figure 4.5), while case 2 is for the daily skewness of 0 (Figure 4.6).

Case 1: Daily skewness $\varsigma = 0.5$

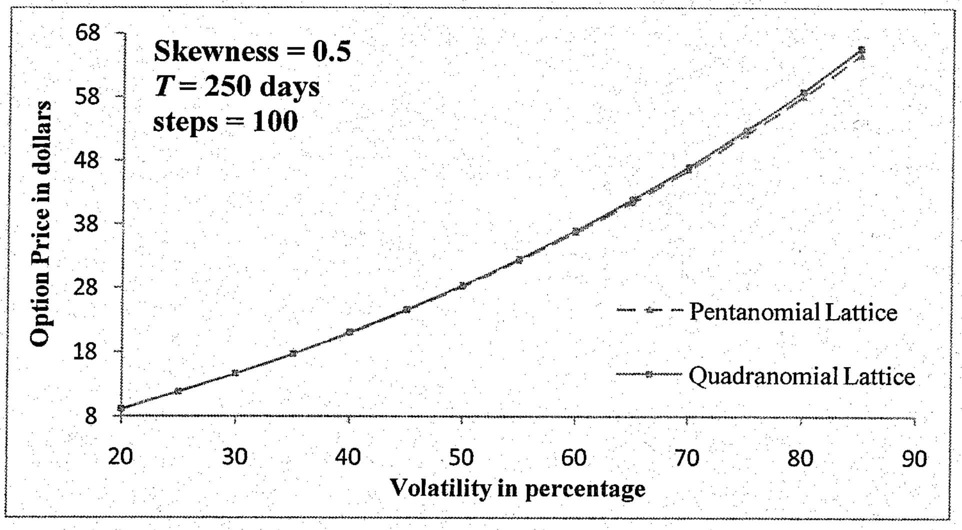


Figure 4.5 Effect of volatility on pentanomial and quadrinomial models for an at-the-money call option with daily skewness equal to 0.5.

Case 2: Daily skewness $\varsigma = 0$

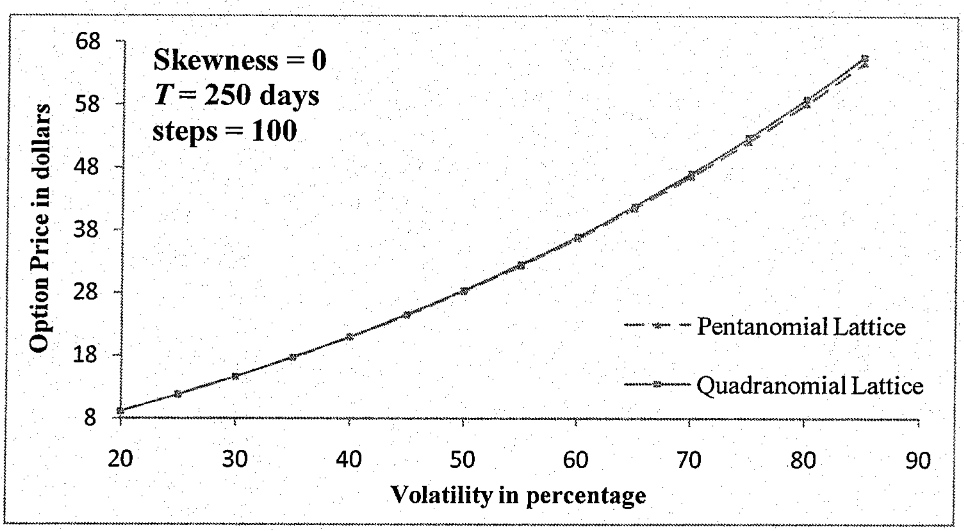


Figure 4.6 Effect of volatility on pentanomial and quadrinomial models for an at-the-money call option with daily skewness equal to 0.

As we can see from Figures 4.5 and 4.6, the quadrinomial lattice model provides the same option price as the pentanomial lattice even for high volatilities.

4.4 Volatility Smiles and Smirks

In real markets, traders usually use the Black-Scholes model but not the same way that it was originated. The Black-Scholes model assumes that the implied volatility is constant and homogeneous for options with different strike prices and maturity periods. But that is not the case. In practice, the implied volatility of options is a function of strike prices and exercise dates. A plot of implied volatility of an option as a function of its strike price is known as a volatility smile (Hull, 2006).

Usually two types of patterns have been observed. One of them shows valley type pattern, known as a volatility smile, whereas the other possesses a skewed smile, known as a volatility skew or volatility smirk. A volatility smile is a pattern in which out-of-money and in-the-money options tend to have higher implied volatilities than at-the-money options. Volatility smiles have usually been noticed in currency markets. Volatility smirks, observed in equity markets, exhibit a slopping pattern in which the implied volatilities of high strike price options are lower than those of at-the-money options. In terms of a probability distribution, the implied distributions having smiles possess heavier tails than the lognormal distributions, while the distributions with smirks are usually skewed with one side having a heavier left tail and a less heavy right tail than the lognormal distributions.

Volatility smiles and smirks are obtained using both the quadrinomial lattice model that is developed in Chapter 3 and the pentanomial lattice model given in Primbs *et al.* (2007) for a European call option using yearly volatility of 0.2, a daily kurtosis of 3, a risk-free rate of 0, and an initial stock price of \$100. For three different maturity periods (20, 50 and 100 days), volatility smiles and smirks are determined when daily skewness values are 0.5 and 0, where

250 steps are used.

As we can see, Figures 4.7, 4.8 and 4.9 show volatility smirks for both the quadrinomial and the pentanomial lattices when daily skewness = 0.5. The implied volatility is slightly skewed and exhibits lower implied volatility for out-of-the money options (high strike price) than that of at-the-money options.

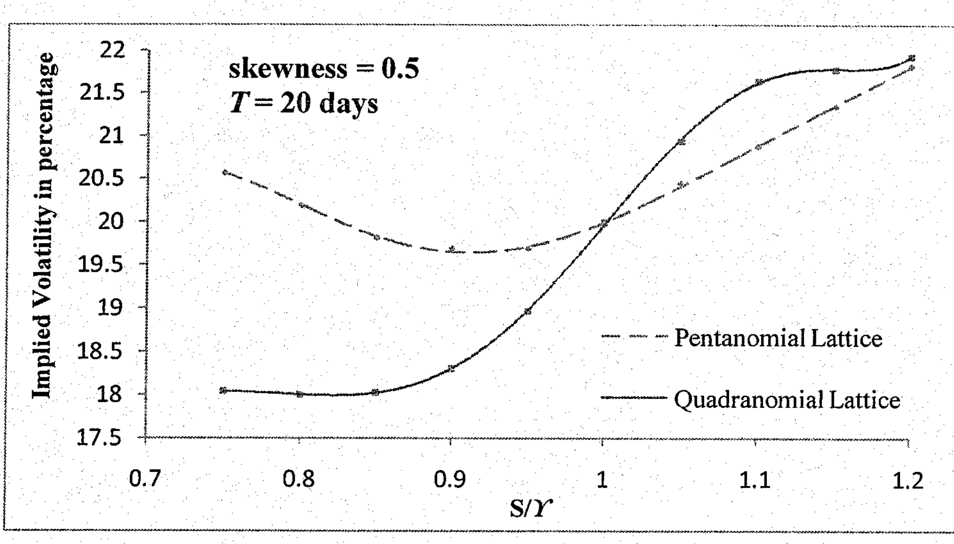


Figure 4.7 Implied volatility plots using pentanomial and quadrinomial models for an at-the-money call option with $\varsigma = 0.5$ and $T = 20$ days.

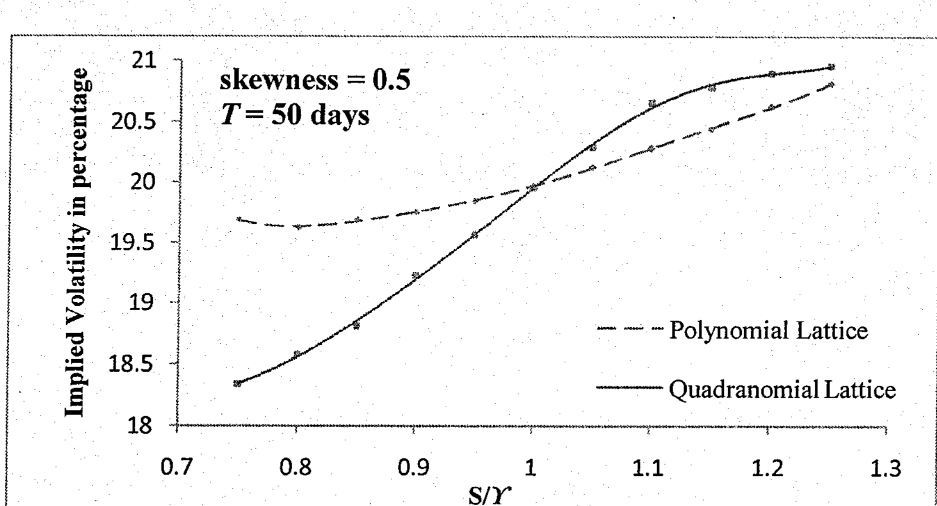


Figure 4.8 Implied volatility plots using pentanomial and quadrinomial models for an at-the-money call option with $\varsigma = 0.5$ and $T = 50$ days.

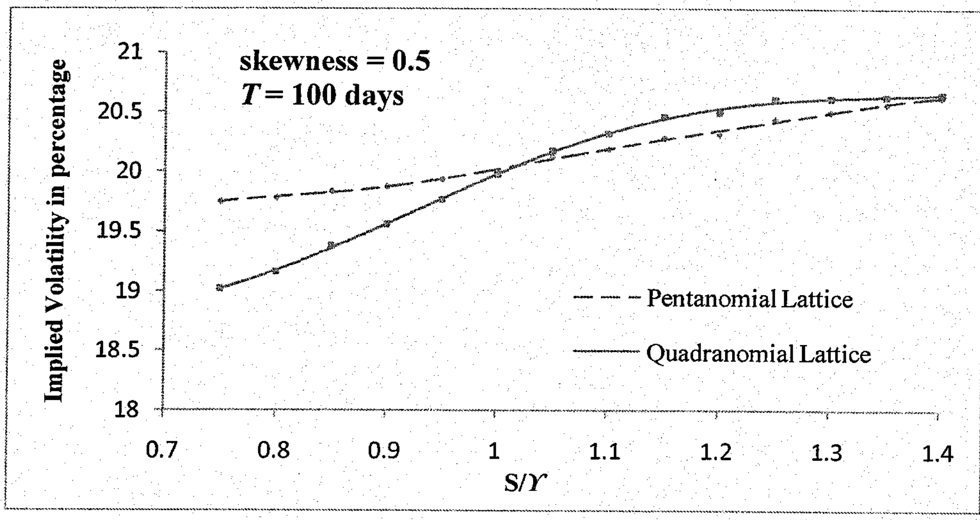


Figure 4.9 Implied volatility plots using pentanomial and quadrinomial models for an at-the-money call option with $\varsigma = 0.5$ and $T = 100$ days.

Figures 4.10, 4.11, and 4.12 are the plots for different maturity periods for skewness 0. The plots possess a valley shaped pattern, which are volatility smiles. The figures show that the quadrinomial lattice model gives almost the same implied volatility as the one obtained using the pentanomial lattice model.

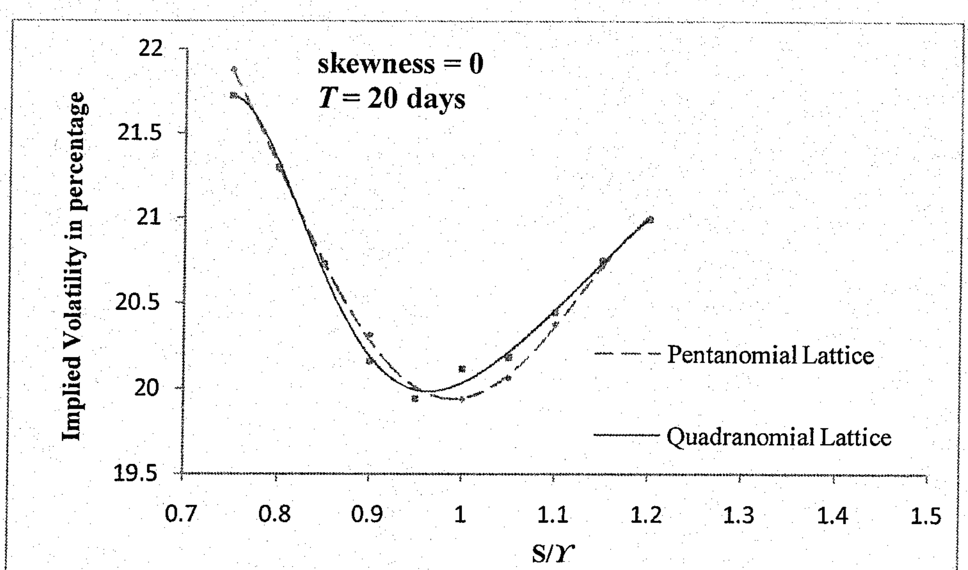


Figure 4.10 Implied volatility plots using pentanomial and quadrinomial models for an at-the-money call option with $\varsigma = 0$ and $T = 20$ days.

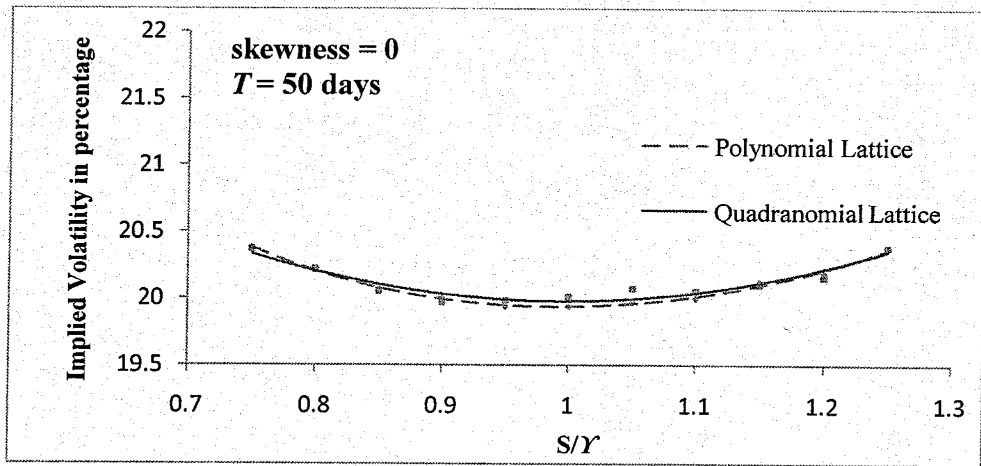


Figure 4.11 Implied volatility plots using pentanomial and quadrinomial models for an at-the-money call option with $\varsigma = 0$ and $T = 50$ days.

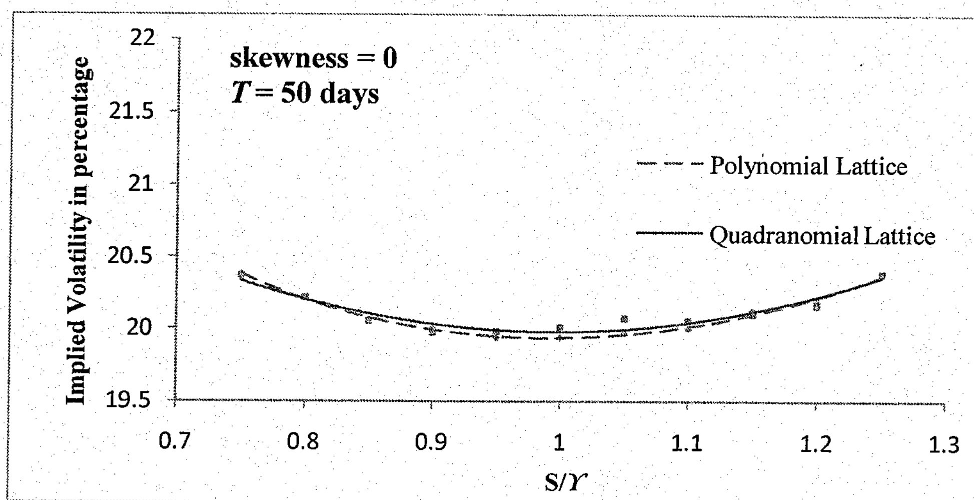


Figure 4.12 Implied volatility plots using pentanomial and quadrinomial models for an at-the-money call option with $\varsigma = 0$ and $T = 100$ days.

Different characteristics of the quadrinomial lattice model have been demonstrated and the figures show that there is not much difference in the characteristics of quadrinomial and pentanomial lattices. Hence, one can use the quadrinomial lattice model for option pricing. The next chapter concludes the results and offers recommendations.

CHAPTER 5

Conclusion and Recommendations

In this research, we have proved that the recombination conditions along with non-negative probabilities for the quadrinomial lattice are the same as the conditions provided for a pentanomial lattice in Primbs *et al.* (2007). This has been done by developing an asymmetric quadrinomial lattice model incorporating skewness and kurtosis along with mean and volatility. In the numerical examples, we have also shown how a quadrinomial lattice model can be used to value options of an underlying variable having skewness and kurtosis along with mean and standard deviation. Furthermore, several characteristics of the quadrinomial lattice have been compared with those of the pentanomial lattice. The convergence rate of quadrinomial lattice is almost the same the convergence rate of pentanomial lattice; and the difference between the calculated option prices is less than 2 cents when we use the time step-size as one day for a European option with 250 days of maturity period. The effect of high volatility values on the quadrinomial lattice is same as that of the pentanomial lattice. Moreover, both the models generate fine volatility smiles and smirks. In short, one can apply the quadrinomial lattice model developed in this research to estimate the option price for a single underlying asset. As for future research, this approach can be extended to develop a multinomial lattice model for multiple underlying assets.

Appendix A

Calculation of the probabilities obtained in Equation (3.8).

We have following five equations:

$$(2\alpha)\, p_1 + (\alpha)\, p_2 + (0)\, p_3 + (-\alpha)\, p_4 = \mu_1 \tag{3.3}$$

$$(2\alpha)^2\, p_1 + (\alpha)^2\, p_2 + (0)^2\, p_3 + (-\alpha)^2\, p_4 = \mu_2 \tag{3.4}$$

$$(2\alpha)^3\, p_1 + (\alpha)^3\, p_2 + (0)^3\, p_3 + (-\alpha)^3\, p_4 = \mu_3 \tag{3.5}$$

$$(2\alpha)^4\, p_1 + (\alpha)^4\, p_2 + (0)^4\, p_3 + (-\alpha)^4\, p_4 = \mu_4 \tag{3.6}$$

$$p_1 + p_2 + p_3 + p_4 = 1 \tag{3.7}$$

Applying the Gaussian Elimination Method,

$$\begin{bmatrix} 4 & 1 & 0 & 1 \\ 8 & 1 & 0 & -1 \\ 16 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} \left(\frac{\mu_2}{\alpha^2}\right) \\ \left(\frac{\mu_3}{\alpha^3}\right) \\ \left(\frac{\mu_4}{\alpha^4}\right) \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & -1 & 0 & -3 \\ 0 & -3 & 0 & -3 \\ 0 & \frac{3}{4} & 1 & \frac{3}{4} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} \left(\frac{\mu_2}{4\alpha^2}\right) \\ \left(\frac{\mu_3 - 2\mu_2\alpha}{\alpha^3}\right) \\ \left(\frac{\mu_4 - 4\mu_2\alpha^2}{\alpha^4}\right) \\ \left(\frac{4\alpha^2 - \mu_2}{4\alpha^2}\right) \end{bmatrix} \qquad \left\{ \begin{gathered} R_1 = \frac{R_1}{4}, \\ R_2 = R_2 - 8R_1, \\ R_3 = R_3 - 16R_1, \\ R_4 = R_4 - R_1 \end{gathered} \right\}$$

$$
\begin{bmatrix} 1 & 0 & 0 & \frac{-1}{2} \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 1 & \frac{-3}{2} \end{bmatrix}
\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}
=
\begin{bmatrix} \left(\frac{\mu_3 - \mu_2\alpha}{4\alpha^3}\right) \\ \left(\frac{-\mu_3 + 2\mu_2\alpha}{\alpha^3}\right) \\ \left(\frac{\mu_4 - 3\mu_3\alpha + 2\mu_2\alpha^2}{\alpha^4}\right) \\ \left(\frac{3\mu_3 - 7\mu_2\alpha + 4\alpha^3}{4\alpha^3}\right) \end{bmatrix}
\quad
\left\{ \begin{array}{c} R_2 = \frac{R_2}{-1}, \\ R_1 = R_1 - \frac{R_2}{4}, \\ R_3 = R_3 + 3R_2, \\ R_4 = R_4 - \frac{3R_2}{4} \end{array} \right\}
$$

$$
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}
\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}
=
\begin{bmatrix} \left(\frac{\mu_4 - \mu_2\alpha^2}{12\alpha^4}\right) \\ \left(\frac{-\mu_4 + \mu_3\alpha + 2\mu_2\alpha^2}{2\alpha^4}\right) \\ \left(\frac{\mu_4 - 3\mu_3\alpha + 2\mu_2\alpha^2}{6\alpha^4}\right) \\ \left(1 + \frac{\mu_4 - 5\mu_2\alpha^2}{4\alpha^4}\right) \end{bmatrix}
\quad
\left\{ \begin{array}{c} R_3 = \frac{R_3}{6}, \\ R_1 = R_1 + \frac{R_3}{2}, \\ R_2 = R_2 - 3R_3, \\ R_4 = R_4 + \frac{3R_3}{2} \end{array} \right\}
$$

Therefore,

$$
\begin{aligned}
p_1 &= \frac{\mu_4 - \mu_2\alpha^2}{12\alpha^4} \\
p_2 &= \frac{-\mu_4 + \mu_3\alpha + 2\mu_2\alpha^2}{2\alpha^4} \\
p_3 &= 1 + \frac{\mu_4 - 5\mu_2\alpha^2}{4\alpha^4} \\
p_4 &= \frac{\mu_4 - 3\mu_3\alpha + 2\mu_2\alpha^2}{6\alpha^4}
\end{aligned}
\tag{3.8}
$$

Appendix B

First four central moments for the lognormal distribution.

Consider a variable, V, follows the lognormal distribution with probability density function $h(V)$. For the lognormal distribution, the n^{th} raw moment, m_n, can be calculated using following equations (Ref. Technical notes 2, Hull 2002),

$$m_n = \int_0^{\infty} V^n h(V)dV = \exp\left(n\varrho + \frac{n\omega^2}{2}\right), \tag{1}$$

where ϱ and ω are the mean and the standard deviation of $ln(S_T)$ given by,

$$\varrho = ln\, S_0 + \left(\vartheta - \frac{\sigma^2}{2}\right)T, \tag{2}$$

$$\omega = \sigma\sqrt{T}\,, \tag{3}$$

and using the relationship between the raw moments and central moments, the first four central moments for the variable V can be estimated.

So, using Equation (1), (2), and (3), the first four moments can be estimated as follows:

For the mean (first moment), $n = 1$.

$$m_1 = \int_0^{\infty} Vh(V)dV = \exp\,(\varrho + \frac{\omega^2}{2})$$

$$= exp\left[ln\, S_0 + \left(\vartheta - \frac{\sigma^2}{2}\right)T + \frac{\sigma^2 T}{2}\right]$$

$$= \exp\,(\ln S_0 + \vartheta T)$$

$$= S_0\, e^{\vartheta T}.$$

Therefore, $\mu_1 = m_1 = S_0\, e^{\vartheta T}$. (4)

For the variance (second moment), $n = 2$.

$$m_2 = \int_0^\infty V^2 h(V)dV = \exp\,(2\varrho + \frac{4\omega^2}{2})$$

$$= exp\left[2\left\{ln\,S_0 + \left(\vartheta - \frac{\sigma^2}{2}\right)T\right\} + 2\sigma^2 T\right]$$

$$= {S_0}^2 e^{2\vartheta T} e^{\sigma^2 T}.$$

Therefore, $\mu_2 = m_2 - {m_1}^2$

$$= {S_0}^2 e^{2\vartheta T} e^{\sigma^2 T} - {S_0}^2 e^{2\vartheta T}$$

$$= {S_0}^2 e^{2\vartheta T}(e^{\sigma^2 T} - 1). \qquad (5)$$

For the skewness (third moment), $n = 3$.

$$m_3 = \int_0^\infty V^3 h(V)dV = \exp\,(3\varrho + \frac{9\omega^2}{2})$$

$$= exp\left[3\left\{ln\,S_0 + \left(\vartheta - \frac{\sigma^2}{2}\right)T\right\} + \frac{9}{2}\sigma^2 T\right]$$

$$= {S_0}^3 e^{3\vartheta T} e^{3\sigma^2 T}.$$

Therefore, $\mu_3 = m_3 - 3m_1 m_2 + 2{m_1}^3$

$$= {S_0}^3 e^{3\vartheta T} e^{3\sigma^2 T} - \left[3\left(S_0\, e^{\vartheta T}\right)\left({S_0}^2 e^{2\vartheta T} e^{\sigma^2 T}\right)\right] + 2{S_0}^3 e^{3\vartheta T}$$

$$= {S_0}^3 e^{3\vartheta T}(e^{3\sigma^2 T} - 3e^{\sigma^2 T} + 2). \qquad (6)$$

For the kurtosis (fourth moment), $n = 4$.

$$m_4 = \int_0^\infty V^4 h(V)dV = \exp\,(4\varrho + \frac{16\omega^2}{2})$$

$$= exp\left[4\left\{ln\,S_0 + \left(\vartheta - \frac{\sigma^2}{2}\right)T\right\} + 8\sigma^2 T\right]$$

$$= {S_0}^4 e^{4\vartheta T} e^{6\sigma^2 T}.$$

Therefore, $\mu_4 = m_4 - 4m_1 m_3 + 6{m_1}^2 m_2 - 3{m_1}^4$

$$= {S_0}^4 e^{4\vartheta T} e^{6\sigma^2 T} - \left[4\left(S_0\, \mathrm{e}^{\vartheta T}\right)\left({S_0}^3 e^{3\vartheta T} e^{3\sigma^2 T}\right)\right]$$

$$+\left[6{S_0}^2 e^{2\vartheta T}\left({S_0}^2 e^{2\vartheta T} e^{\sigma^2 T}\right)\right] - 3{S_0}^4 e^{4\vartheta T}$$

$$= {S_0}^4 e^{4\vartheta T}\left(e^{6\sigma^2 T} - 4e^{3\sigma^2 T} + 6e^{\sigma^2 T} - 3\right). \tag{7}$$

Appendix C

Derivation of the conditions for the negatively skewed lattice shown in Figure 3.2 (b):

For the negatively skewed lattice, consider the jump size,

$$\alpha = \frac{-\mu_3 - \sqrt{{\mu_3}^2 + \mu_2\mu_4}}{\mu_2}. \tag{1}$$

In terms of cumulants, the above equation can be written as,

$$\alpha = \frac{-c_3 - \sqrt{{c_3}^2 + c_2(c_4 + 3{c_2}^2\tau)}}{c_2}. \tag{2}$$

As $\tau \to 0$, we have,

$$\alpha_0 = \frac{-c_3 - \sqrt{{c_3}^2 + c_2c_4}}{c_2}. \tag{3}$$

Now we have the probability distributions (Equation (3.23)) as follows:

$$\begin{aligned}
\lim_{\tau\to 0}\left(\frac{1}{\tau}\right)p_1(\tau) &= \frac{c_4 - c_2{\alpha_0}^2}{12{\alpha_0}^4} = \xi q_1,\\
\lim_{\tau\to 0}\left(\frac{1}{\tau}\right)p_2(\tau) &= \frac{-c_4 + c_3\alpha_0 + 2c_2{\alpha_0}^2}{2{\alpha_0}^4} = \xi q_2,\\
\lim_{\tau\to 0}\left(\frac{1}{\tau}\right)(p_3(\tau) - 1) &= \frac{c_4 - 5c_2{\alpha_0}^2}{4{\alpha_0}^4} = -\xi\,,\\
\lim_{\tau\to 0}\left(\frac{1}{\tau}\right)p_4(\tau) &= \frac{c_4 - 3c_3\alpha_0 + 2c_2{\alpha_0}^2}{6{\alpha_0}^4} = \xi q_4.
\end{aligned} \tag{4}$$

Substituting Equation (3) into Equation (4), and solving for $\xi, q_1, q_2,$ and q_4, we have,

$$
\begin{aligned}
\xi &= \frac{c_2{}^3\left(5c_3{}^2 + 2c_2c_4 + 5c_3\sqrt{c_3{}^2 + c_2c_4}\right)}{2\left(c_3 + \sqrt{c_3{}^2 + c_2c_4}\right)^4}, \\
q_1 &= \frac{-c_3\left(c_3 + \sqrt{c_3{}^2 + c_2c_4}\right)}{3\left(5c_3{}^2 + 2c_2c_4 + 5c_3\sqrt{c_3{}^2 + c_2c_4}\right)}, \\
q_2 &= \frac{3c_3{}^2 + c_2c_4 + 3c_3\sqrt{c_3{}^2 + c_2c_4}}{5c_3{}^2 + 2c_2c_4 + 5c_3\sqrt{c_3{}^2 + c_2c_4}}, \\
q_4 &= \frac{7c_3{}^2 + 3c_2c_4 + 7c_3\sqrt{c_3{}^2 + c_2c_4}}{15c_3{}^2 + 6c_2c_4 + 15c_3\sqrt{c_3{}^2 + c_2c_4}}.
\end{aligned}
\tag{5}
$$

From Equation (5), q_1, q_2, and q_4 can also be written in terms of ξ, that is,

$$
\begin{aligned}
\xi &= \frac{c_2{}^3}{2}\left[\frac{2c_2c_4}{\left(\sqrt{c_3{}^2 + c_2c_4} + c_3\right)^4} + \frac{5c_3}{\left(\sqrt{c_3{}^2 + c_2c_4} + c_3\right)^3}\right], \\
q_1 &= \frac{c_2{}^3}{6\xi}\left[\frac{-c_3}{\left(\sqrt{c_3{}^2 + c_2c_4} + c_3\right)^3}\right], \\
q_2 &= \frac{c_2{}^3}{6\xi}\left[\frac{3c_2c_4}{\left(\sqrt{c_3{}^2 + c_2c_4} + c_3\right)^4} + \frac{9c_3}{\left(\sqrt{c_3{}^2 + c_2c_4} + c_3\right)^3}\right], \\
q_4 &= \frac{c_2{}^3}{6\xi}\left[\frac{3c_2c_4}{\left(\sqrt{c_3{}^2 + c_2c_4} + c_3\right)^4} + \frac{7c_3}{\left(\sqrt{c_3{}^2 + c_2c_4} + c_3\right)^3}\right].
\end{aligned}
\tag{6}
$$

In Equation (5), for q_1 to be positive,

$$c_3 \leq 0 \tag{7}$$

is the first condition since negatively skewed lattice has been considered.

Moreover,

$$\sqrt{c_3{}^2 + c_2c_4} + c_3 \geq 0. \tag{8}$$

Therefore,

$$\sqrt{c_3{}^2 + c_2c_4} \geq -c_3. \tag{9}$$

Since, according to Equation (7), $c_3 \leq 0$, the right side of the Equation (9) becomes positive, so simplifying Equation (9), we get,

$$c_4 \geq 0. \tag{10}$$

Now, for ξ to be positive,

$$\frac{2c_2c_4}{\left(\sqrt{c_3^2 + c_2c_4} + c_3\right)^4} + \frac{5c_3}{\left(\sqrt{c_3^2 + c_2c_4} + c_3\right)^3} \geq 0. \tag{11}$$

$$\frac{2c_2c_4}{\left(\sqrt{c_3^2 + c_2c_4} + c_3\right)^4} \geq \frac{-5c_3}{\left(\sqrt{c_3^2 + c_2c_4} + c_3\right)^3}. \tag{12}$$

$$2c_2c_4 \geq -5c_3\left(\sqrt{c_3^2 + c_2c_4}\right) - 5c_3^2. \tag{13}$$

Therefore,

$$2c_2c_4 + 5c_3^2 \geq -5c_3\left(\sqrt{c_3^2 + c_2c_4}\right). \tag{14}$$

Since $c_3 \leq 0$ according to Equation (7), the right side of the Equation (14) becomes positive, so simplifying Equation (14), we get,

$$c_2c_4 \geq \left(\frac{5}{4}\right)c_3^2. \tag{15}$$

For q_2 to be positive:

$$\frac{3c_2c_4}{\left(\sqrt{c_3^2 + c_2c_4} + c_3\right)^4} \geq \frac{-9c_3}{\left(\sqrt{c_3^2 + c_2c_4} + c_3\right)^3}. \tag{16}$$

With the same argument that is used for Equation (14), i.e., since we have $c_3 \leq 0$, the right side of the Equation (16) becomes positive. Hence, by simplifying Equation (16), we get,

$$c_2c_4 \geq 3c_3^2. \tag{17}$$

Finally, for q_4 to remain positive:

$$\frac{3c_2c_4}{\left(\sqrt{c_3{}^2 + c_2c_4} + c_3\right)^4} \geq \frac{-7c_3}{\left(\sqrt{c_3{}^2 + c_2c_4} + c_3\right)^3}, \tag{18}$$

which leads to,

$$c_2c_4 \geq \left(\frac{7}{9}\right) c_3{}^2. \tag{19}$$

Among all the conditions given in Equations (15), (17), and (19), the most constraining condition is,

$$c_2c_4 \geq 3c_3{}^2.$$

Therefore, there are three conditions that must be satisfied for non-negative probability distributions for the negatively skewed lattice:

$$c_2c_4 \geq 3c_3{}^2, \qquad c_4 \geq 0, \quad \text{and} \quad c_3 \leq 0\,. \tag{20}$$

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